VTU EDUSAT PROGRAMME-17

DYNAMICS OF MACHINES
Subject Code -10 ME 54

BALANCING OF RECIPROCATING MASSES

Notes Compiled by:
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SLIDER CRANK MECHANISM:

PRIMARY AND SECONDARY ACCELERATING FORCE:

Acceleration of the reciprocating mass of a slider-crank mechanism is given by,

$$a_p = \text{Acceleration of piston}$$
$$= r\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right] ----------(1)$$
$$\text{Where } n = \frac{l}{r}$$

And, the force required to accelerate the mass 'm' is

$$F_i = mr\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$
$$= mr\omega^2\cos\theta + mr\omega^2\frac{\cos 2\theta}{n} ----------(2)$$

The first term of the equation (2) , i.e. $mr\omega^2 \cos\theta$ is called **primary accelerating force** the second term $mr\omega^2 \frac{\cos 2\theta}{n}$ is called the **secondary accelerating force.**

Maximum value of primary accelerating force is $mr\omega^2$

And Maximum value of secondary accelerating force is $\frac{mr\omega^2}{n}$

Generally, 'n' value is much greater than one; the secondary force is small compared to primary force and can be safely neglected for slow speed engines.

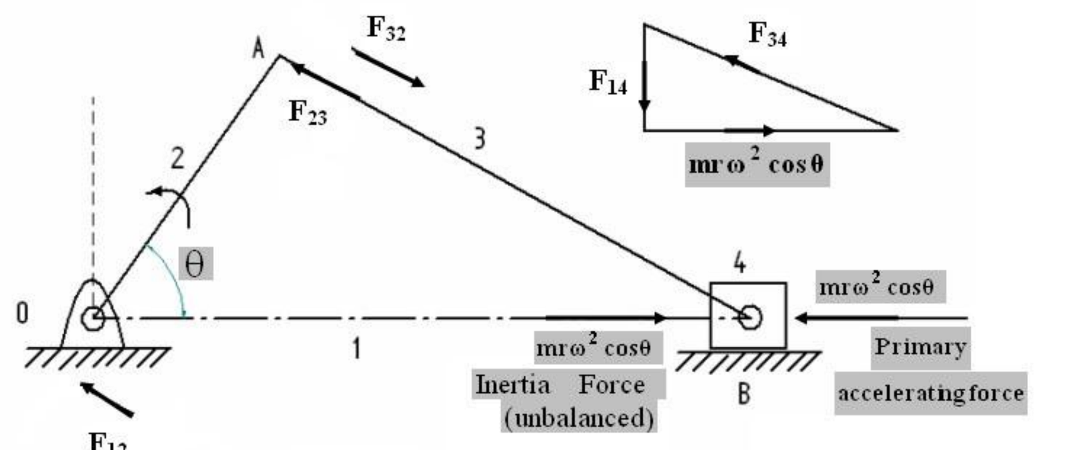


Figure (a)

In Fig (a), the inertia force due to primary accelerating force is shown.

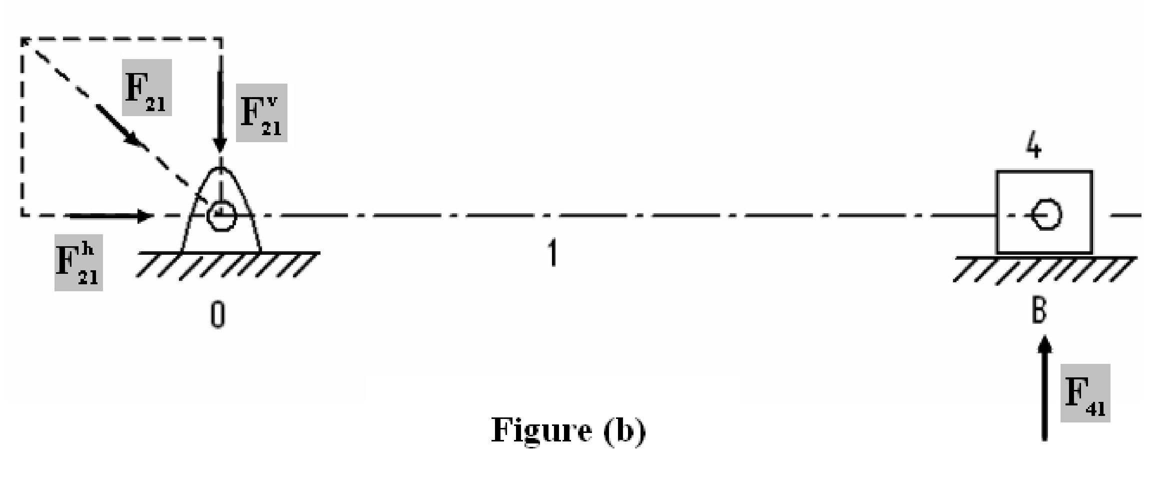


Figure (b)

In Fig (b), the forces acting on the engine frame due to inertia force are shown.

At ‘O’ the force exerted by the crankshaft on the main bearings has two components, horizontal $\mathbf{F}_{21}^{\mathbf{h}}$ and vertical $\mathbf{F}_{21}^{\mathbf{v}}$.

$\mathbf{F}_{21}^{\mathbf{h}}$ is an horizontal force, which is an **unbalanced shaking force.**

$\mathbf{F}_{21}^{\mathbf{v}}$ and $\mathbf{F}_{41}^{\mathbf{v}}$ balance each other but form an **unbalanced shaking couple**.

The magnitude and direction of these unbalanced force and couple go on changing with angle θ. The shaking force produces linear vibrations of the frame in horizontal direction, whereas the shaking couple produces an oscillating vibration.

The shaking force $\mathbf{F}_{21}^{\mathbf{h}}$ is the only unbalanced force which may hamper the smooth running of the engine and effort is made to balance the same.
However it is not at all possible to balance it completely and only some modifications can be carried out.

BALANCING OF THE SHAKING FORCE:

Shaking force is being balanced by adding a rotating counter mass at radius ‘r’ directly opposite the crank. This provides only a partial balance. This counter mass is in addition to the mass used to balance the rotating unbalance due to the mass at the crank pin. This is shown in figure (c).

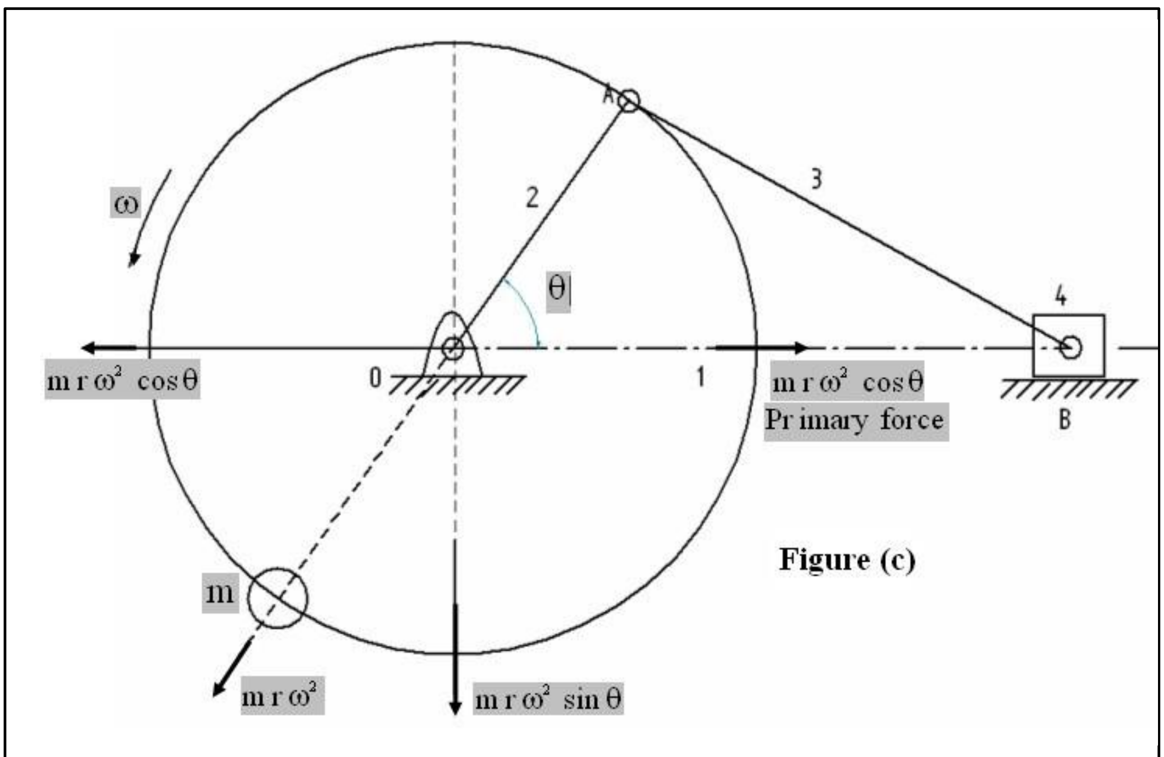


Figure (c)

The horizontal component of the centrifugal force due to the balancing mass is $\mathbf{mr}\,\omega^2\,\mathbf{cos}\theta$ and this is in the line of stroke. This component neutralizes the unbalanced reciprocating force. But the rotating mass also has a component $\mathbf{mr}\omega^2\,\mathbf{sin}\theta$ perpendicular to the line of stroke which remains unbalanced. The unbalanced force is zero at $\theta = 0^0$ or 180^0 and maximum at the middle of the stroke i.e. $\theta = 90^0$. The magnitude or the maximum value of the unbalanced force remains the same i.e. equal to $\mathbf{mr}\,\omega^2$. Thus instead of sliding to and fro on its mounting, the mechanism tends to jump up and down.

To minimize the effect of the unbalance force a compromise is, usually made, is $\frac{2}{3}$ of the reciprocating mass is balanced or a value between $\frac{1}{2}$ to $\frac{3}{4}$.

If ‘c’ is the fraction of the reciprocating mass, then

$$\text{The primary force balanced by the mass} = c\,m\,r\,\omega^2\cos\theta$$

and

$$\text{The primary force unbalanced by the mass} = (1-c)\,m\,r\,\omega^2\cos\theta$$

$$\text{Vertical component of centrifugal force which remains unbalanced}$$
$$= c\,m\,r\,\omega^2\sin\theta$$

In reciprocating engines, unbalance forces in the direction of the line of stroke are more dangerous than the forces perpendicular to the line of stroke.

$$\text{Resultant unbalanced force at any instant}$$
$$= \sqrt{\left[(1-c)\,m\,r\,\omega^2\cos\theta\right]^2 + \left[c\,m\,r\,\omega^2\sin\theta\right]^2}$$

The resultant unbalanced force is minimum when, $\mathbf{c} = \frac{1}{2}$

This method is just equivalent to as if a revolving mass at the crankpin is completely balanced by providing a counter mass at the same radius diametrically opposite to the crank. Thus if $\mathbf{m_P}$ is the mass at the crankpin and ‘c’ is the fraction of the reciprocating mass ‘m’ to be balanced , the mass at the crankpin may be considered as $\mathbf{cm}+\mathbf{m_P}$ which is to be completely balanced.

Problem 1:
A single –cylinder reciprocating engine has a reciprocating mass of 60 kg. The crank rotates at 60 rpm and the stroke is 320 mm. The mass of the revolving parts at 160 mm radius is 40 kg. If two-thirds of the reciprocating parts and the whole of the revolving parts are to be balanced, determine the, (i) balance mass required at a radius of 350 mm and (ii) unbalanced force when the crank has turned 50^0 from the top-dead centre.

Solution:

$$\text{Given: } m = \text{mass of the reciprocating parts} = 60 \text{ kg}$$
$$N = 60 \text{ rpm}, \; L = \text{length of the stroke} = 320 \text{ mm}$$
$$m_P = 40 \text{ kg}, \; c = \frac{2}{3}, \; r_c = 350 \text{ mm}$$

(i) Balance mass required at a radius of 350 mm

We have, $\omega = \frac{2\pi N}{60} = \frac{2\pi x 60}{60} = 2\pi \text{ rad/s}$

$$r = \frac{L}{2} = \frac{320}{2} = 160 \text{ mm}$$

$$\text{Mass to be balanced at the crank pin} = M$$
$$M = c\,m + m_P = \frac{2}{3} x 60 + 40 = 80 \text{ kg}$$

and $m_c\, r_c = M r \quad \text{therefore} \quad m_c = \frac{M r}{r_c}$

$$\text{i.e. } m_c = \frac{80 x 160}{350} = 36.57 \text{ kg}$$

(ii) Unbalanced force when the crank has turned 50^0 from the top-dead centre.

$$\text{Unbalanced force at } \theta = 50^0$$
$$= \sqrt{\left[(1-c) m r \omega^2 \cos\theta\right]^2 + \left[c m r \omega^2 \sin\theta\right]^2}$$
$$= \sqrt{\left[\left(1 - \frac{2}{3}\right) x 60 x 0.16 x (2\pi)^2 \cos 50^0\right]^2 + \left[\frac{2}{3} x 60 x 0.16 x (2\pi)^2 \sin 50^0\right]^2}$$
$$= 209.9 \text{ N}$$

Problem 2:

The following data relate to a single cylinder reciprocating engine:
Mass of reciprocating parts = 40 kg
Mass of revolving parts = 30 kg at crank radius
Speed = 150 rpm, Stroke = 350 mm.
If 60 % of the reciprocating parts and all the revolving parts are to be balanced, determine the,
(i) balance mass required at a radius of 320 mm and (ii) unbalanced force when the crank has turned 45^0 from the top-dead centre.

Solution:

$$\text{Given:} \quad m = \text{mass of the reciprocating parts} = 40 \text{ kg}$$
$$m_P = 30 \text{ kg}, \; N = 150 \text{ rpm}, \; L = \text{length of the stroke} = 350 \text{ mm}$$
$$c = 60\%, \; r_c = 320 \text{ mm}$$

(i) Balance mass required at a radius of 350 mm

We have,
$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 150}{60} = 15.7 \text{ rad/s}$$
$$r = \frac{L}{2} = \frac{350}{2} = 175 \text{ mm}$$

$$\text{Mass to be balanced at the crank pin} = M$$
$$M = c\,m + m_P = 0.60 \times 40 + 30 = 54 \text{ kg}$$

and
$$m_c r_c = M r \quad \text{therefore} \quad m_c = \frac{M r}{r_c}$$
$$\text{i.e. } m_c = \frac{54 \times 175}{320} = 29.53 \text{ kg}$$

(ii) Unbalanced force when the crank has turned 45^0 from the top-dead centre.

$$\text{Unbalanced force at } \theta = 45^0$$
$$= \sqrt{\left[(1-c) m r \omega^2 \cos\theta\right]^2 + \left[c m r \omega^2 \sin\theta\right]^2}$$
$$= \sqrt{\left[(1-0.60) \times 40 \times 0.175 \times (15.7)^2 \cos 45^0\right]^2 + \left[0.60 \times 40 \times 0.175 \times (15.7)^2 \sin 45^0\right]^2}$$
$$= 880.7 \text{ N}$$

SECONDARY BALANCING:

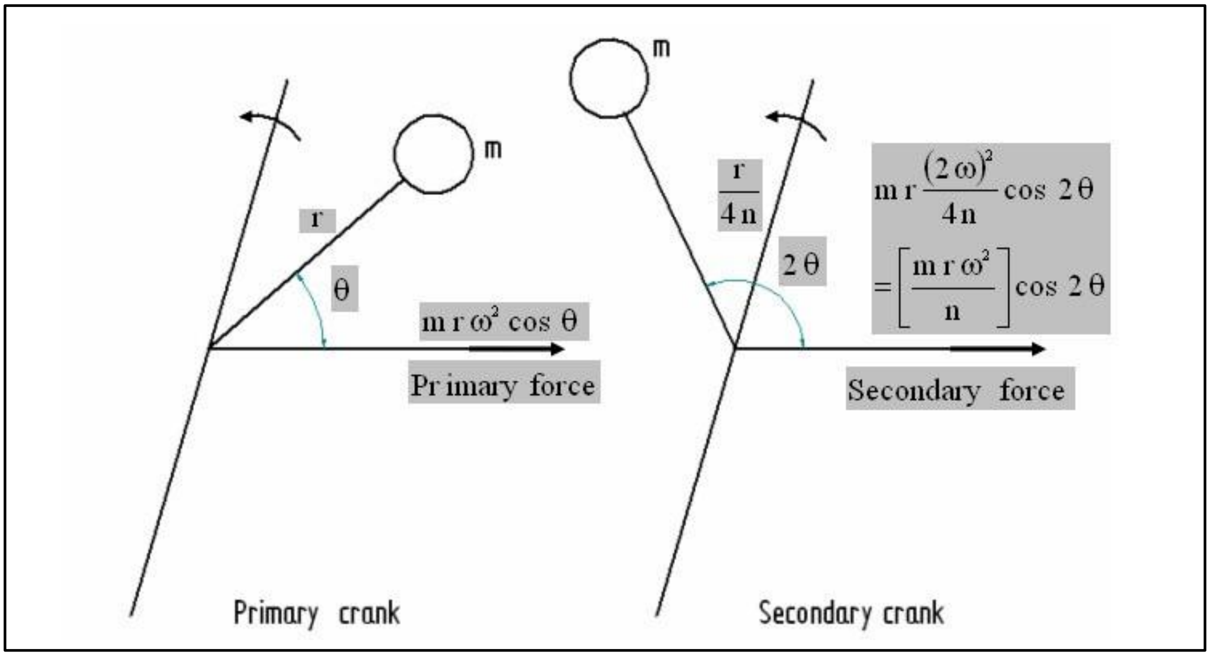
Secondary acceleration force is equal to $mr\omega^2 \dfrac{\cos 2\theta}{n}$ ––––––(1)

Its frequency is twice that of the primary force and the magnitude $1/n$ times the magnitude of the primary force.

The secondary force is also equal to $mr(2\omega)^2 \dfrac{\cos 2\theta}{4n}$ ––––––(2)

Consider, two cranks of an engine, one actual one and the other imaginary with the following specifications.

	Actual	Imaginary
Angular velocity	ω	2ω
Length of crank	r	$\dfrac{r}{4n}$
Mass at the crank pin	m	m



Thus, when the actual crank has turned through an angle $\theta = \omega t$, the imaginary crank would have turned an angle $2\theta = 2\omega t$

Centrifugal force induced in the imaginary crank = $\frac{mr(2\omega)^2}{4n}$

Component of this force along the line of stroke is = $\frac{mr(2\omega)^2}{4n}\cos 2\theta$

Thus the effect of the secondary force is equivalent to an imaginary crank of length $\frac{r}{4n}$ rotating at double the angular velocity, i.e. twice of the engine speed. The imaginary crank coincides with the actual at inner top-dead centre. At other times, it makes an angle with the line of stroke equal to twice that of the engine crank.
The secondary couple about a reference plane is given by the multiplication of the secondary force with the distance ‘ l ’ of the plane from the reference plane.

COMPLETE BALANCING OF RECIPROCATING PARTS

Conditions to be fulfilled:
1. Primary forces must balance i.e., primary force polygon is enclosed.
2. Primary couples must balance i.e., primary couple polygon is enclosed.
3. Secondary forces must balance i.e., secondary force polygon is enclosed.
4. Secondary couples must balance i.e., secondary couple polygon is enclosed.

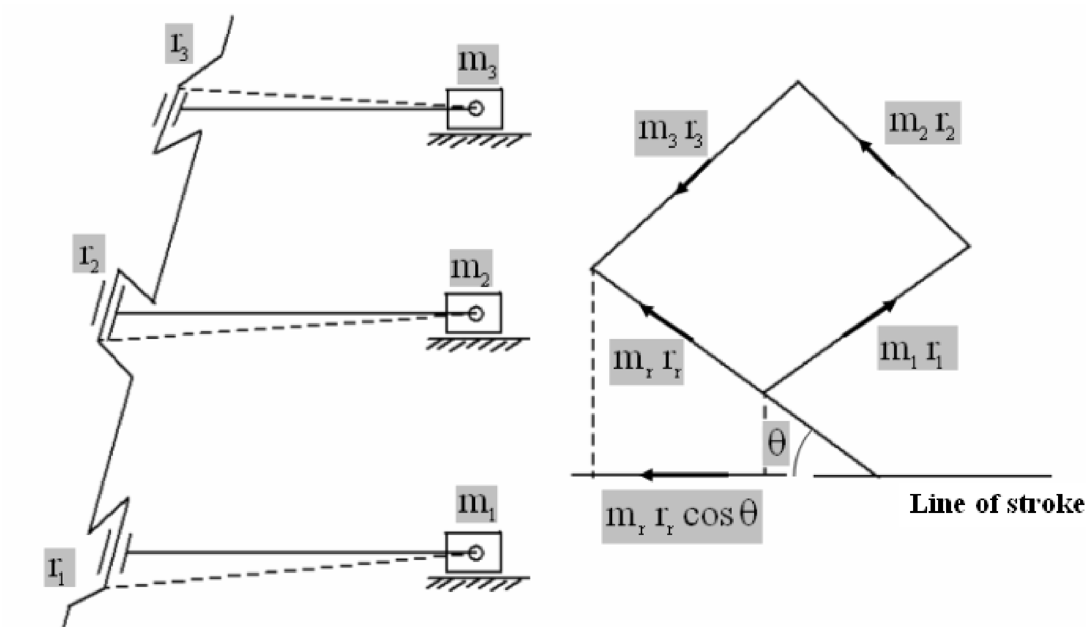
Usually, it is not possible to satisfy all the above conditions fully for multi-cylinder engine. Mostly some unbalanced force or couple would exist in the reciprocating engines.

BALANCING OF INLINE ENGINES:

An in-line engine is one wherein all the cylinders are arranged in a single line, one behind the other. Many of the passenger cars such as Maruti 800, Zen, Santro, Honda-city, Honda CR-V, Toyota corolla are the examples having four cinder in-line engines.

In a reciprocating engine, the reciprocating mass is transferred to the crankpin; the axial component of the resulting centrifugal force parallel to the axis of the cylinder is the primary unbalanced force.

Consider a shaft consisting of three equal cranks asymmetrically spaced. The crankpins carry equivalent of three unequal reciprocating masses, then



BALANCING OF INLINE ENGINES

$$\text{Primary force} = \sum m r \omega^2 \cos\theta \text{------------------}(1)$$

$$\text{Primary couple} = \sum m r \omega^2 l \cos\theta \text{------------------}(2)$$

$$\text{Secondary force} = \sum m r \frac{(2\omega)^2}{4n} \cos 2\theta \text{------------------}(3)$$

$$\text{And Secondary couple} = \sum m r \frac{(2\omega)^2}{4n} l \cos 2\theta$$

$$= \sum m r \frac{\omega^2}{n} l \cos 2\theta \text{------------------}(4)$$

GRAPHICAL SOLUTION:

To solve the above equations graphically, first draw the $\sum m r \cos\theta$ polygon (ω^2 is common to all forces). Then the axial component of the resultant forces $(\mathbf{F_r \cos\theta})$ multiplied by ω^2 provides the primary unbalanced force on the system at that moment. This unbalanced force is zero when $\theta = 90^0$ and a maximum when $\theta = 0^0$.

If the force polygon encloses, the resultant as well as the axial component will always be zero and the system will be in **primary balance**.
Then,

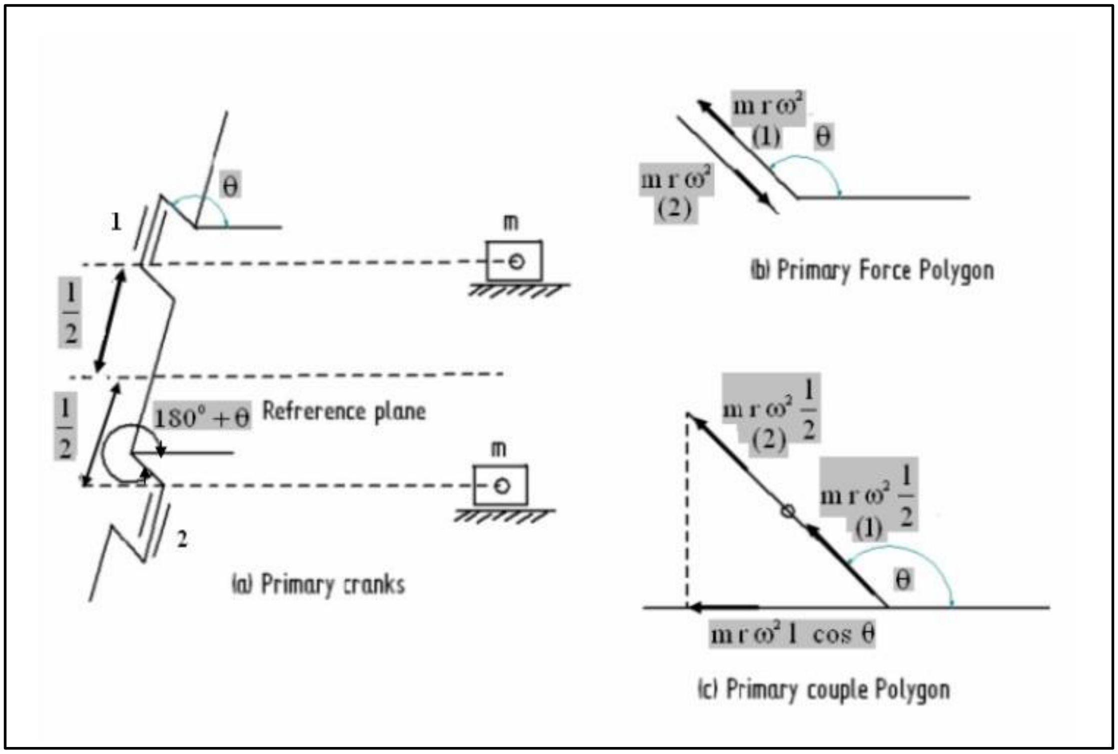
$$\sum F_{Ph} = 0 \text{ and } \sum F_{PV} = 0$$

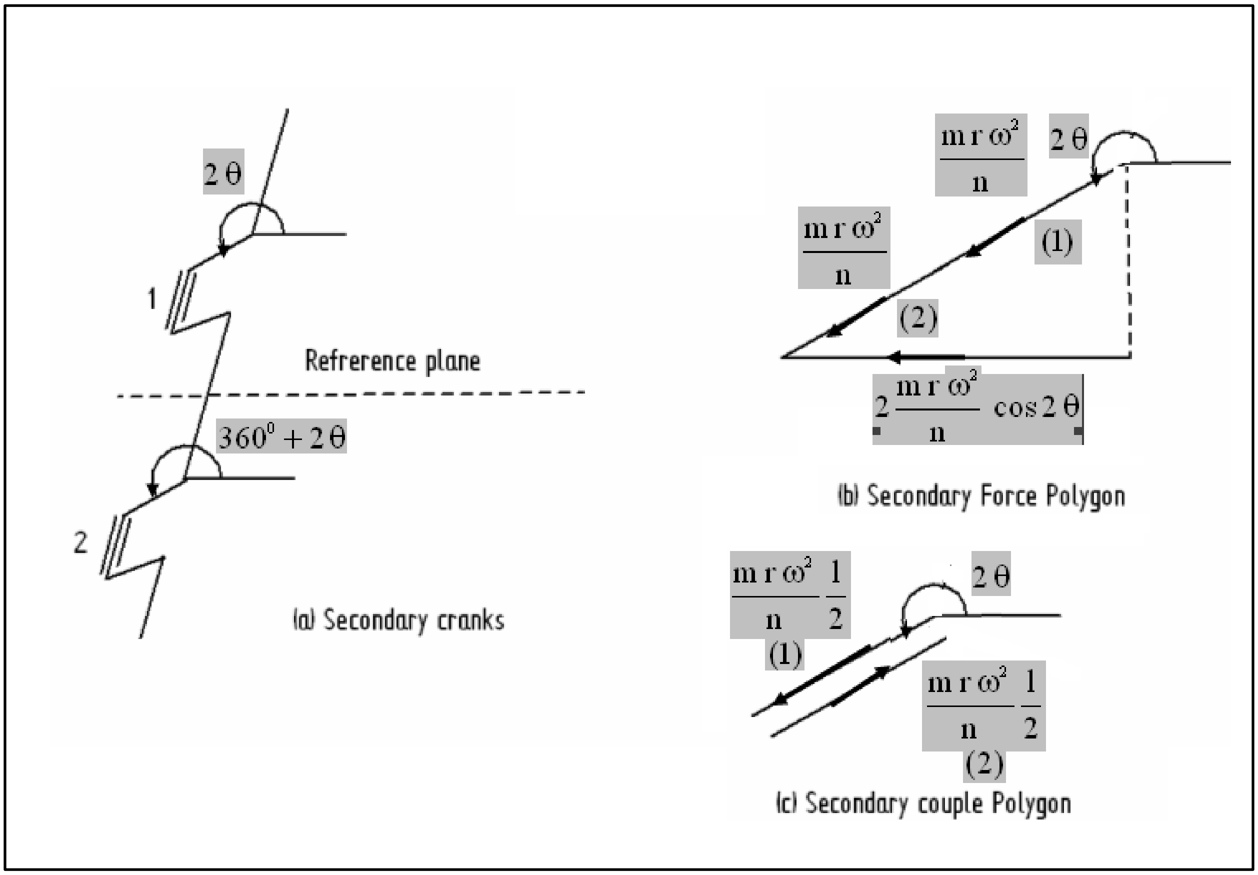
To find the secondary unbalance force, first find the positions of the imaginary secondary cranks. Then transfer the reciprocating masses and multiply the same by $\frac{(2\omega)^2}{4n}$ or $\frac{\omega^2}{n}$ to get the secondary force.
In the same way primary and secondary couple (m r l) polygon can be drawn for primary and secondary couples.

Case 1:

IN-LINE TWO-CYLINDER ENGINE

Two-cylinder engine, cranks are 180^0 apart and have equal reciprocating masses.





Taking a plane through the **centre line** as the reference plane,

$$\text{Primary force} = m r \omega^2 \left[\cos\theta + \cos(180 + \theta)\right] = 0$$

$$\text{Primary couple} = m r \omega^2 \left[\frac{l}{2}\cos\theta + \left(-\frac{l}{2}\right)\cos(180 + \theta)\right] = m r \omega^2 l \cos\theta$$

Maximum values are $m r \omega^2 l$ at $\theta = 0^0$ and 180^0

$$\text{Secondary force} = \frac{m r \omega^2}{n}\left[\cos 2\theta + \cos(360 + 2\theta)\right] = \frac{2 m r \omega^2}{n}\cos 2\theta$$

Maximum values are $\dfrac{2 m r \omega^2}{n}$ when $2\theta = 0^0, 180^0, 360^0$ and 540^0
or $\theta = 0^0, 90^0, 180^0$ and 270^0

$$\text{Secondary couple} = \frac{m r \omega^2}{n}\left[\frac{l}{2}\cos 2\theta + \left(-\frac{l}{2}\right)\cos(360 + 2\theta)\right] = 0$$

ANALYTICAL METHOD OF FINDING PRIMARY FORCES AND COUPLES

- First the positions of the cranks have to be taken in terms of θ^0
- The maximum values of these forces and couples vary instant to instant and are equal to the values as given by the equivalent rotating masses at the crank pin.

If a particular position of the crank shaft is considered, the above expressions may not give the maximum values.
For example, the maximum value of primary couple is $m r \omega^2 l$ and this value is obtained at crank positions 0^0 and 180^0. However, if the crank positions are assumed at 90^0 and 270^0, the values obtained will be zero.

- If any particular position of the crank shaft is considered, then both X and Y components of the force and couple can be taken to find the maximum values.

For example, if the crank positions considered as 120^0 and 300^0, the primary couple can be obtained as

$$X - \text{component} = m r \omega^2\left[\frac{l}{2}\cos 120^0 + \left(-\frac{l}{2}\right)\cos\left(180^0 + 120^0\right)\right]$$

$$= -\frac{1}{2} m r \omega^2 l$$

$$Y - \text{component} = m r \omega^2\left[\frac{l}{2}\sin 120^0 + \left(-\frac{l}{2}\right)\sin\left(180^0 + 120^0\right)\right]$$

$$= \frac{\sqrt{3}}{2} m r \omega^2 l$$

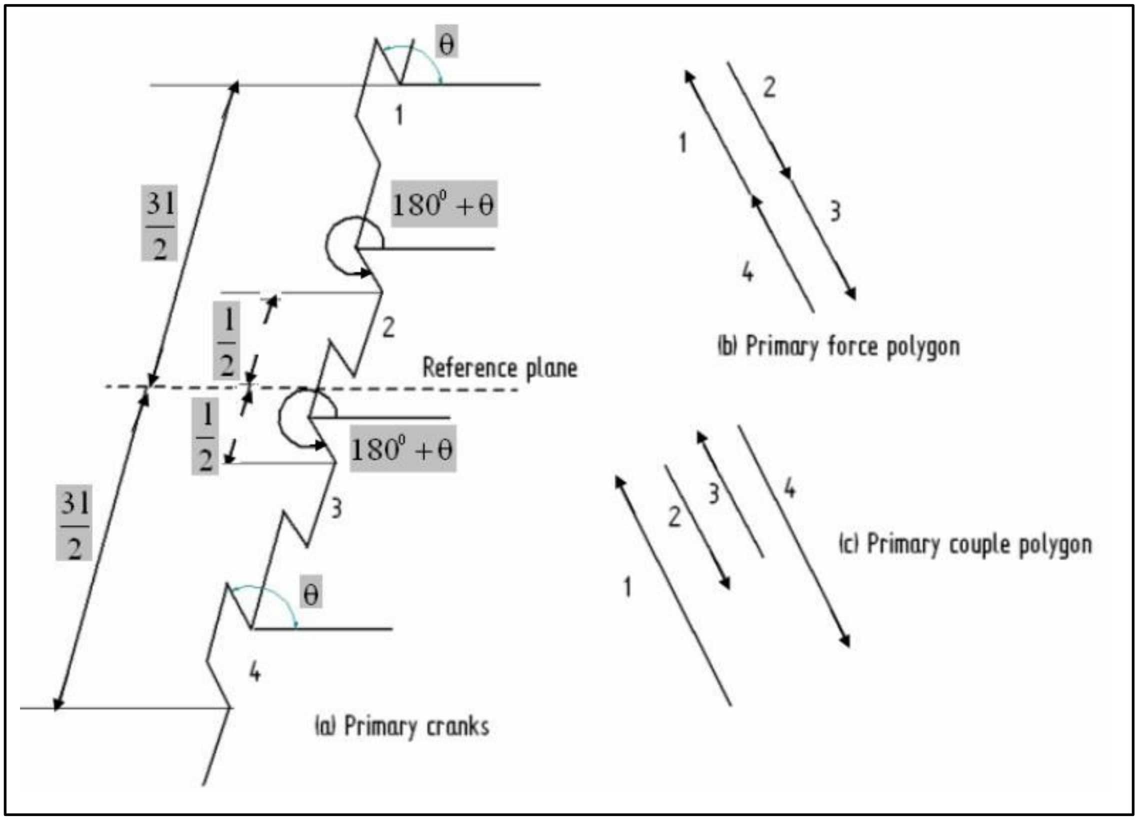
Therefore, $$\text{Primary couple} = \sqrt{\left(-\frac{1}{2} m r \omega^2 l\right)^2 + \left(\frac{\sqrt{3}}{2} m r \omega^2 l\right)^2}$$

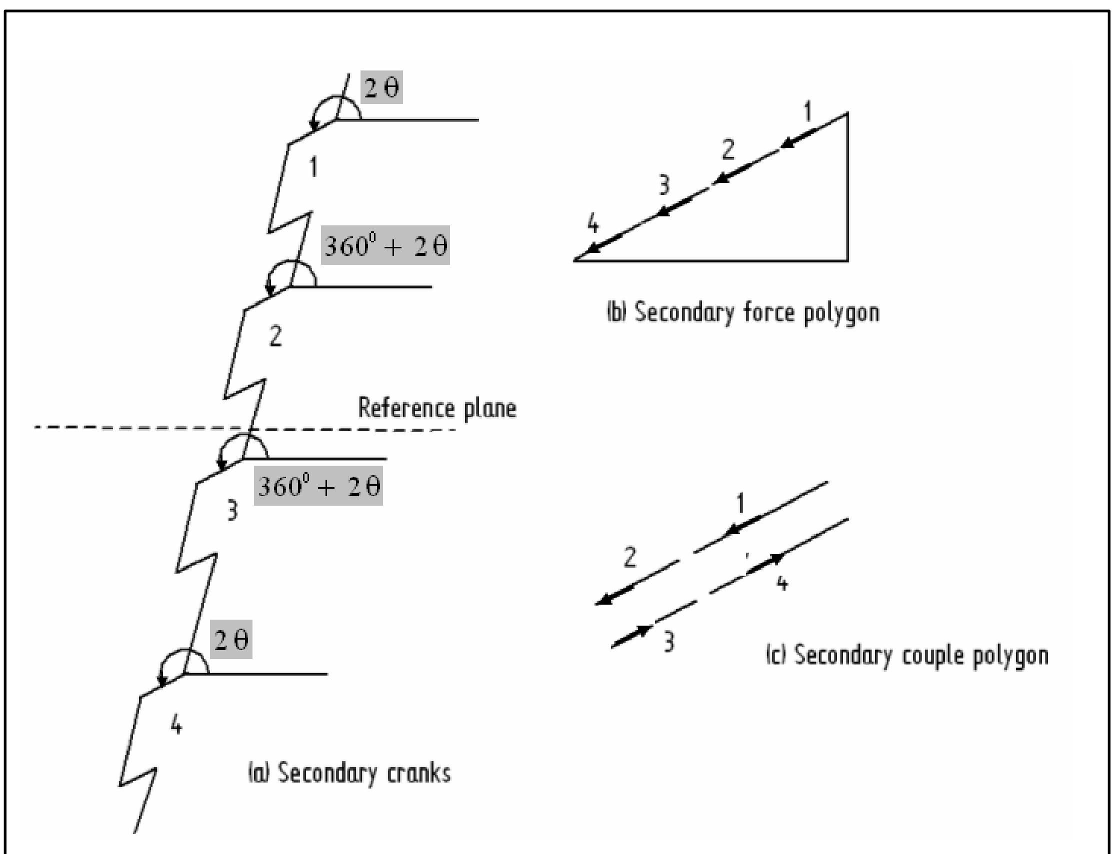
$$= m r \omega^2 l$$

Case 2:

IN-LINE FOUR-CYLINDER FOUR-STROKE ENGINE

This engine has tow outer as well as inner cranks (throws) in line. The inner throws are at 180^0 to the outer throws. Thus the angular positions for the cranks are θ° for the first, $180^\circ + \theta^\circ$ for the second, $180^\circ + \theta^\circ$ for the third and θ° for the fourth.





FINDING PRIMARY FORCES, PRIMARY COUPLES, SECONDARY FORCES AND SECONDARY COUPLES:

Choose a plane passing through the middle bearing about which the arrangement is symmetrical as the reference plane.

$$\text{Primary force} = m r \omega^2 \left[\cos\theta + \cos(180^0 + \theta) + \cos(180^0 + \theta) + \cos\theta\right]$$
$$= 0$$

$$\text{Primary couple} = m r \omega^2 \left[\begin{array}{l} \frac{3l}{2}\cos\theta + \frac{l}{2}\cos(180^0 + \theta) \\ + \left(-\frac{l}{2}\right)\cos(180^0 + \theta) + \left(-\frac{3l}{2}\right)\cos\theta \end{array}\right]$$
$$= 0$$

$$\text{Secondary force} = \frac{m r \omega^2}{n}\left[\begin{array}{l} \cos 2\theta + \cos(360^0 + 2\theta) \\ + \cos(360^0 + 2\theta) + \cos 2\theta \end{array}\right]$$
$$= \frac{4 m r \omega^2}{n}\cos 2\theta$$

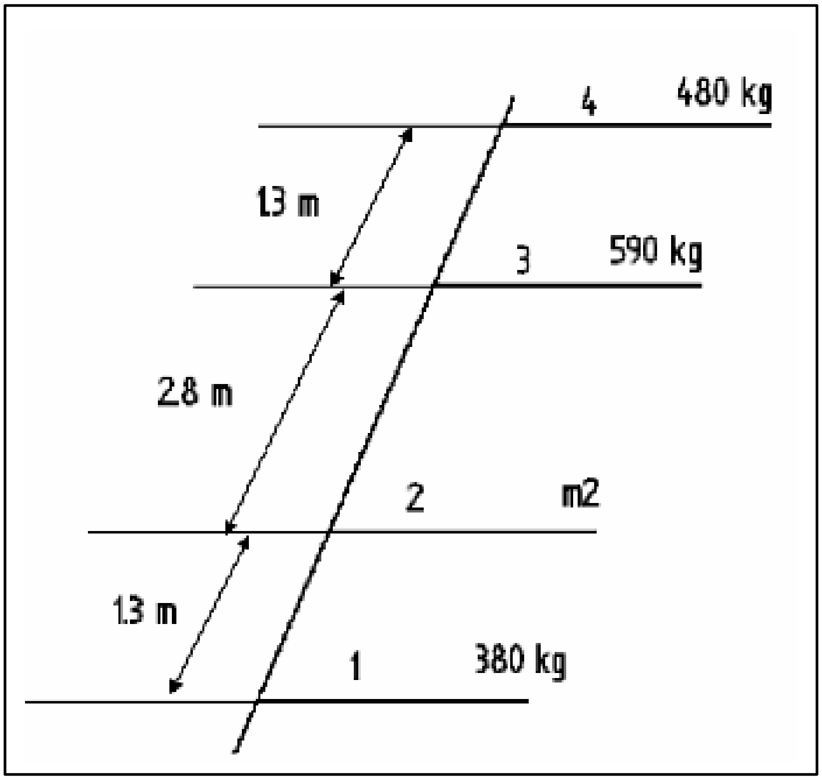
$$\text{Maximum value} = \frac{m r \omega^2}{n}$$
$$\text{at } 2\theta = 0^0, 180^0, 360^0 \text{ and } 540^0 \text{ or}$$
$$\theta = 0^0, 90^0, 180^0 \text{ and } 270^0$$

$$\text{Secondary couple} = \frac{m r \omega^2}{n}\left[\begin{array}{l} \frac{3l}{2}\cos 2\theta + \frac{l}{2}\cos(360^0 + 2\theta) \\ + \left(-\frac{l}{2}\right)\cos(360^0 + 2\theta) + \left(-\frac{3l}{2}\right)\cos 2\theta \end{array}\right] = 0$$

Thus the engine is not balanced in secondary forces.

Problem 1:

A four-cylinder oil engine is in complete primary balance. The arrangement of the reciprocating masses in different planes is as shown in figure. The stroke of each piston is 2 r mm. Determine the reciprocating mass of the cylinder 2 and the relative crank position.



Solution:

Given:

$m_1 = 380$ kg, $m_2 = ?$, $m_3 = 590$ kg, $m_4 = 480$ kg

$$\text{crank length} = \frac{L}{2} = \frac{2r}{2} = r$$

Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane '2' m	Couple/ ω^2 (m r l) kg m^2
1	**380**	**r**	**380 r**	**-1.3**	**-494 r**
2(RP)	**m_2**	**r**	**m_2 r**	**0**	**0**
3	**590**	**r**	**590 r**	**2.8**	**1652 r**
4	**480**	**r**	**480 r**	**4.1**	**1968 r**

Analytical Method:

Choose plane 2 as the reference plane and $\theta_3 = 0^0$.

.

Step 1:

Resolve the couples into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$$-494\,\mathbf{r}\,\mathbf{cos}\,\theta_1 + 1652\,\mathbf{r}\,\mathbf{cos}\,0^0 + 1968\,\mathbf{r}\,\mathbf{cos}\,\theta_4 = 0$$

$$\textbf{i.e.,}\; +494\;\mathbf{cos}\,\theta_1 = 1652 + 1968\;\mathbf{cos}\,\theta_4 ---------- \mathbf{(1)}$$

Sum of the vertical components gives

$$-494\,\mathbf{r}\,\mathbf{sin}\,\theta_1 + 1652\,\mathbf{r}\,\mathbf{sin}\,0^0 + 1968\,\mathbf{r}\,\mathbf{sin}\,\theta_4 = 0$$

$$\textbf{i.e.,}\; 494\,\mathbf{sin}\,\theta_1 = 1968\;\mathbf{sin}\,\theta_4 ---------- \mathbf{(2)}$$

Squaring and adding (1) and (2), we get

$$(494)^2 = (1652 + 1968\cos\theta_4)^2 + (1968\sin\theta_4)^2$$

i.e.,

$$(494)^2 = (1652)^2 + 2 \times 1652 \times 1968 \cos\theta_4 + (1968\cos\theta_4)^2 + (1968\sin\theta_4)^2$$

On solving we get,

$\cos\theta_4 = -0.978$ and $\theta_4 = 167.9^0$ or 192.1^0

Choosing one value, say $\theta_4 = 167.9^0$

Dividing (2) by (1), we get

$$\tan\theta_1 = \frac{1968\sin(167.9^0)}{1652 + 1968\cos(167.9^0)} = \frac{+412.53}{-272.28} = -1.515$$

i.e., $\theta_1 = 123.4^0$

Step 2:

Resolve the forces into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$$380\;\mathbf{r}\,\mathbf{cos}(123.4^0) + \mathbf{m}_2\,\mathbf{r}\,\mathbf{cos}\,\theta_2 + 590\,\mathbf{r}\,\mathbf{cos}\,0^0 + 480\,\mathbf{r}\,\mathbf{cos}(167.9^0) = 0$$

$$\textbf{or}\;\; \mathbf{m}_2\;\mathbf{cos}\,\theta_2 = 88.5 -------------- \mathbf{(3)}$$

Sum of the vertical components gives

$$380\ \mathbf{r}\ \mathbf{sin}(123.4^0) + \mathbf{m}_2\, \mathbf{r}\ \mathbf{sin}\ \theta_2 + 590\ \mathbf{r}\,\mathbf{sin}\, 0^0 + 480\,\mathbf{r}\,\mathbf{sin}(167.9^0) = 0$$

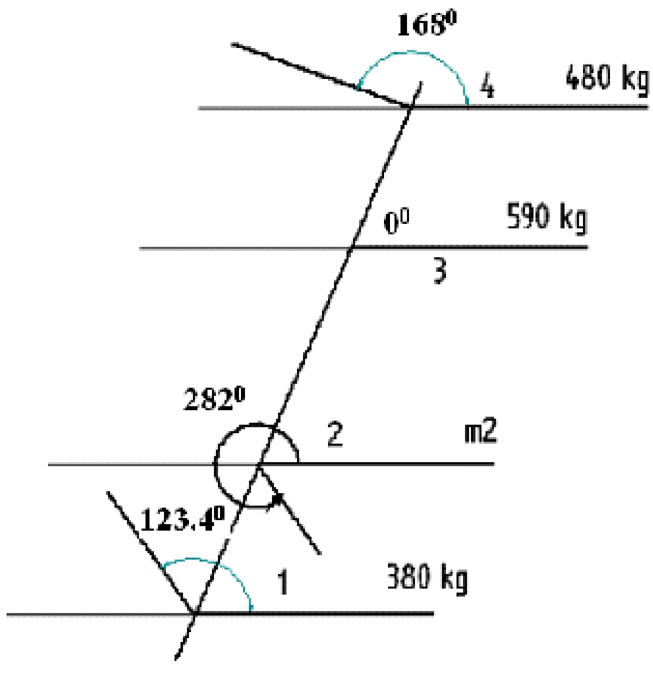
$$\mathbf{or}\ \ \mathbf{m}_2\ \mathbf{sin}\ \theta_2 = -417.9 - - - - - - - - - - - - - - - (4)$$

Squaring and adding (3) and (4), we get

$$\mathbf{m}_2 = 427.1\ \mathbf{kg}\ \ \mathbf{Ans}$$

Dividing (4) by (3), we get $\tan\theta_2 = \dfrac{-417.9}{88.5} = -4.72$

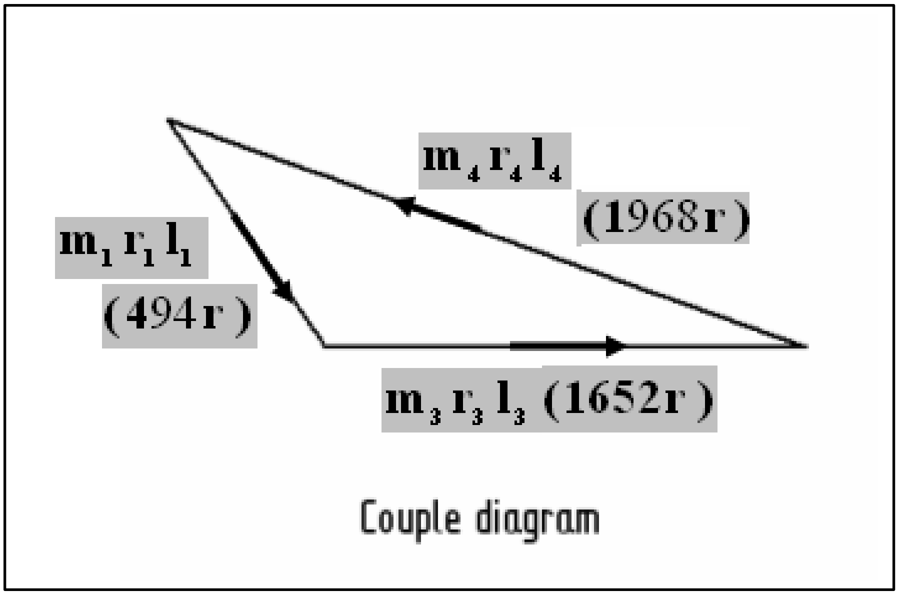
$\text{or}\quad \theta_2 = 282^0$ **Ans**



Relative crank positions

Graphical Method:

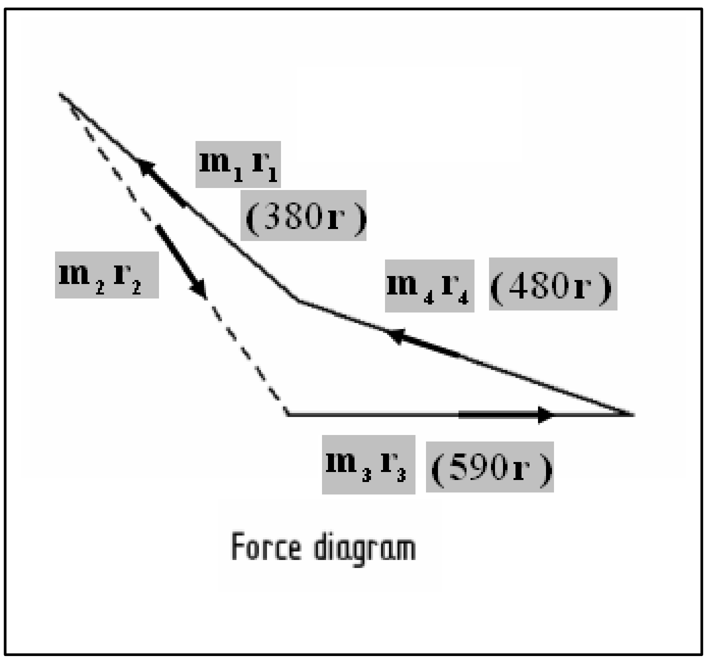
Step 1: Draw the couple diagram taking a suitable scale as shown.



Couple diagram

This diagram provides the relative direction of the masses m_1, m_3 and m_4.

Step 2: Now, draw the force polygon taking a suitable scale as shown.



Force diagram

This gives the direction and magnitude of mass m_2.

The results are:

$$\theta_4 = 168^0,\ \theta_1 = 123^0,\ \theta_2 = 282^0$$

$$\mathbf{m_2 r = 427r \ \ or\ m_2 = 427\ kg\ \ Ans}$$

Problem 2:

Each crank of a four- cylinder vertical engine is 225 mm. The reciprocating masses of the first, second and fourth cranks are 100 kg, 120 kg and 100 kg and the planes of rotation are 600 mm, 300 mm and 300 mm from the plane of rotation of the third crank. Determine the mass of the reciprocating parts of the third cylinder and the relative angular positions of the cranks if the engine is in complete primary balance.

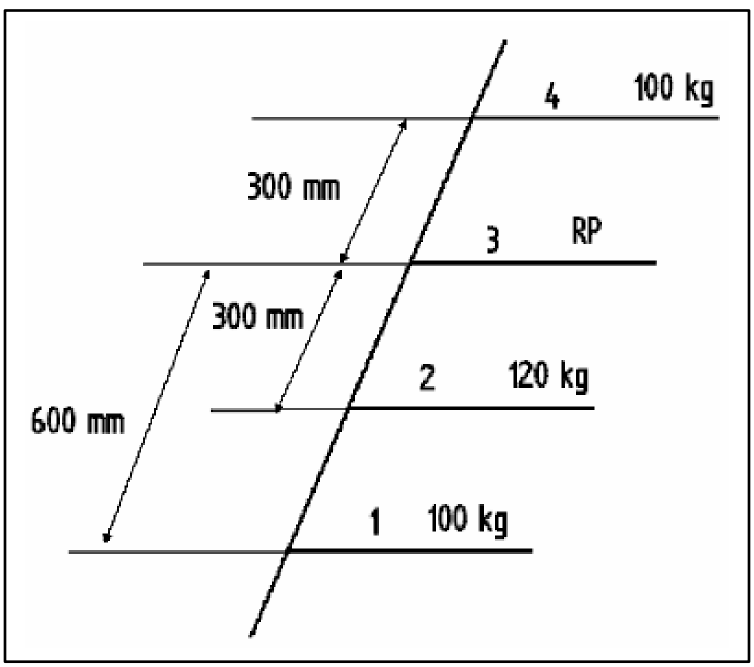
Solution:

Given:

$r = 225$ mm

$m_1 = 100$ kg, $m_2 = 120$ kg and $m_4 = 100$ kg

Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane '2' m	Couple/ ω^2 (m r l) kg m^2
1	100	0.225	22.5	-0.600	-13.5
2	120	0.225	27.0	-0.300	-8.1
3(RP)	m_3	0.225	0.225 m_3	0	0
4	100	0.225	22.5	0.300	6.75



Analytical Method:

Choose plane 3 as the reference plane and $\theta_1 = 0^0$.
Step 1:
Resolve the couples into their horizontal and vertical components and take their sums.
Sum of the horizontal components gives

$$-13.5 \cos 0^0 - 8.1 \cos \theta_2 + 6.75 \cos \theta_4 = 0$$
$$\text{i.e., } -8.1 \cos \theta_2 = -6.75 \cos \theta_4 + 13.5$$
$$\text{i.e., } \quad 8.1 \cos \theta_2 = 6.75 \cos \theta_4 - 13.5 ----------(1)$$

Sum of the vertical components gives

$$-13.5 \sin 0^0 - 8.1 \sin \theta_2 + 6.75 \sin \theta_4 = 0$$
$$\text{i.e., } \quad 8.1 \sin \theta_2 = 6.75 \sin \theta_4 ----------(2)$$

Squaring and adding (1) and (2), we get

$$(8.1)^2 = (6.75 \cos \theta_4 - 13.5)^2 + (6.75 \sin \theta_4)^2$$
$$65.61 = 45.563 \cos^2 \theta_4 - 182.25 \cos \theta_4 + 182.25 + 45.563 \sin^2 \theta_4$$
$$= 45.563(\cos^2 \theta_4 + \sin^2 \theta_4) - 182.25 \cos \theta_4 + 182.25$$
$$= 45.563 - 182.25 \cos \theta_4 + 182.25$$

$$\text{i.e., } 182.25 \cos \theta_4 = 45.563 + 182.25 - 65.61 = 162.203$$

Therefore, $\cos \theta_4 = \dfrac{162.203}{182.25}$ and $\theta_4 = 27.13^0$ Ans

Dividing (2) by (1), we get

$$\tan \theta_2 = \frac{6.75 \sin(27.13^0)}{6.75 \cos(27.13^0) - 13.5} = \frac{3.078}{-7.493} = -1.515$$

$$\text{i.e., } \theta_2 = -22.33^0 + 180^0 = 157.67^0$$

Step 2:
Resolve the forces into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$$22.5\cos(0^0) + 27\cos(157.67^0) + 0.225m_3\cos\theta_3 + 22.5\cos(27.13^0) = 0$$

$$\text{i.e.,}\quad 22.5 - 24.975 + 0.225m_3\cos\theta_3 + 20.02 = 0$$

$$\text{i.e.,}\quad 0.225m_3\cos\theta_3 = -17.545 \text{ ----------------(3)}$$

And sum of the vertical components gives

$$22.5\sin(0^0) + 27\sin(157.67^0) + 0.225m_3\sin\theta_3 + 22.5\sin(27.13^0) = 0$$

$$\text{i.e.,}\quad 10.258 + 0.225m_3\sin\theta_3 + 10.26 = 0$$

$$\text{i.e.,}\quad 0.225m_3\sin\theta_3 = --20.518 \text{ ----------------(4)}$$

Squaring and adding (3) and (4), we get

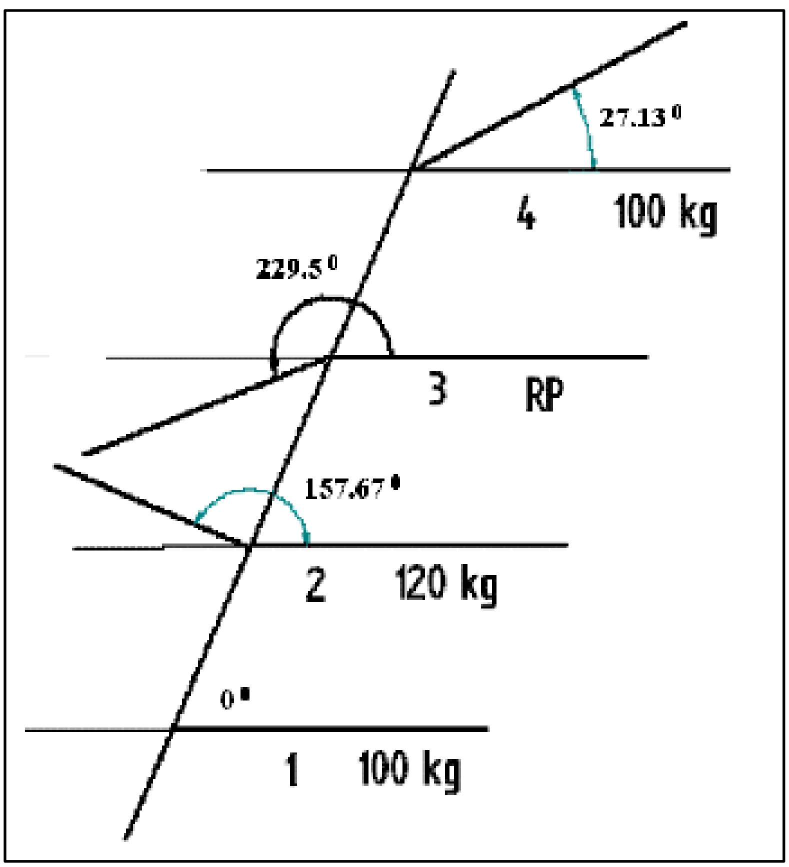
$$(0.225)^2 m_3^2 = (-17.545)^2 + (-20.518)^2$$

$$\text{i.e.,}\quad m_3 = \sqrt{\left(\frac{-17.545}{0.225}\right)^2 + \left(\frac{-20.518}{0.225}\right)^2}$$

$$= 119.98\text{ kg} \approx 120\text{ kg}\quad \text{Ans}$$

Dividing (4) by (3), we get $\tan\theta_3 = \dfrac{-20.518}{-17.545}$

$$\text{or}\quad \theta_3 = 229.5^0\quad \text{Ans}$$



Problem 3:

The cranks of a four cylinder marine oil engine are at angular intervals of 90^0. The engine speed is 70 rpm and the reciprocating mass per cylinder is 800 kg. The inner cranks are 1 m apart and are symmetrically arranged between outer cranks which are 2.6 m apart. Each crank is 400 mm long.
Determine the firing order of the cylinders for the best balance of reciprocating masses and also the magnitude of the unbalanced primary couple for that arrangement.

Analytical Solution:

Given:

$m = 800$ kg, $N = 70$ rpm , $r = 0.4$ m, $\omega = \frac{2\pi N}{60} = 7.33$ rad/s

$m r \omega^2 = 800 \times 0.4 \times (7.33)^2 = 17195$

Note:

There are four cranks. They can be used in six different arrangements as shown. It can be observed that in all the cases, primary forces are always balanced. Primary couples in each case will be as under.

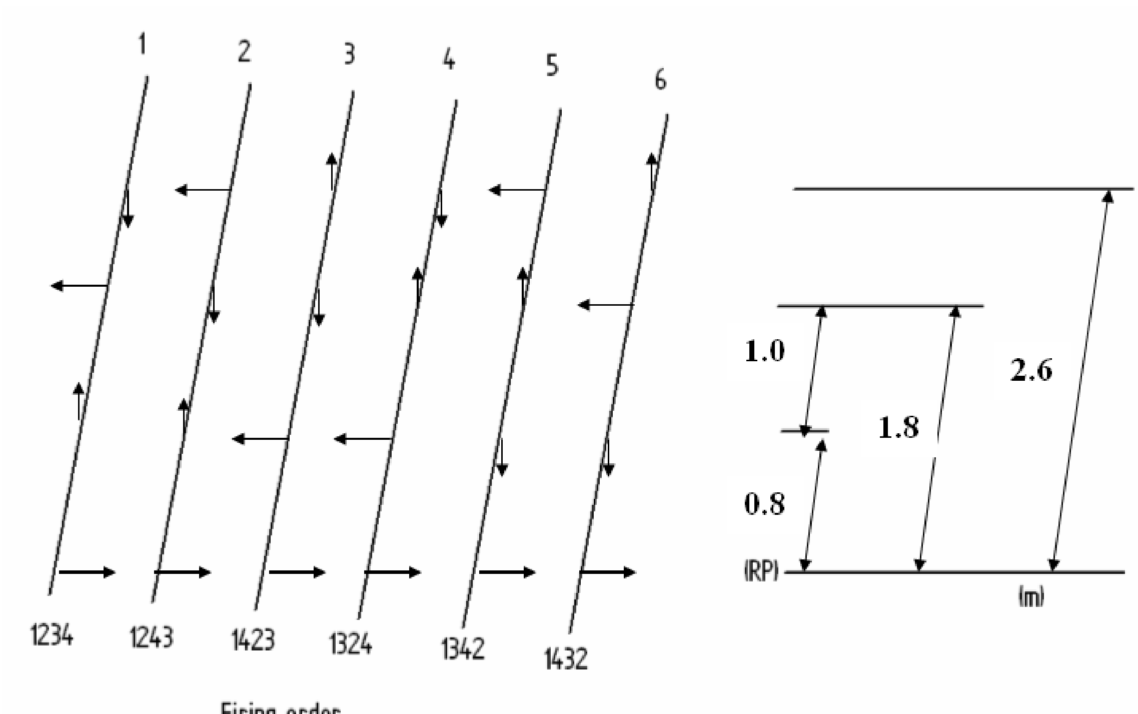


Figure (a)

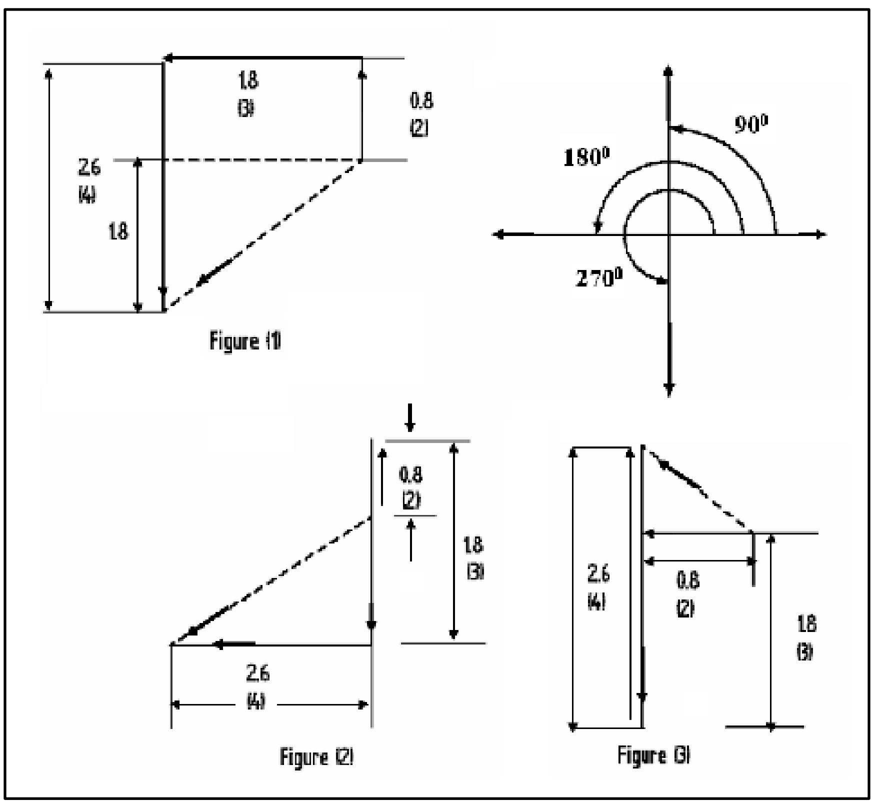
$$\mathbf{C_{p1}} = \mathbf{m\,r}\,\omega^2 \sqrt{(-\mathbf{l_3})^2 + (\mathbf{l_2} - \mathbf{l_4})^2} = 17195\sqrt{(-1.8)^2 + (0.8 - 2.6)^2}$$
$$= 43761 \ \mathbf{Nm}$$
$$\mathbf{C_{p6}} = \mathbf{C_{p1}} = 43761 \ \mathbf{Nm\ only,\ since\ l_2\ and\ l_4\ are\ interchanged}$$

$$\mathbf{C_{p2}} = \mathbf{m\,r}\,\omega^2 \sqrt{(-\mathbf{l_4})^2 + (\mathbf{l_2} - \mathbf{l_3})^2} = 17195\sqrt{(-2.6)^2 + (0.8 - 1.8)^2}$$
$$= 47905 \ \mathbf{Nm}$$
$$\mathbf{C_{p5}} = \mathbf{C_{p2}} = 47905 \ \mathbf{Nm\ only,\ since\ l_2\ and\ l_3\ are\ interchanged}$$

$$\mathbf{C_{p3}} = \mathbf{m\,r}\,\omega^2 \sqrt{(-\mathbf{l_2})^2 + (\mathbf{l_4} - \mathbf{l_3})^2} = 17195\sqrt{(-0.8)^2 + (2.6 - 1.8)^2}$$
$$= 19448 \ \mathbf{Nm}$$
$$\mathbf{C_{p4}} = \mathbf{C_{p3}} = 19448 \ \mathbf{Nm\ only,\ since\ l_4\ and\ l_3\ are\ interchanged}$$

Thus the best arrangement is of 3rd and 4th. The firing orders are 1423 and 1324 respectively.
Unbalanced couple = 19448 N m.

Graphical solution:



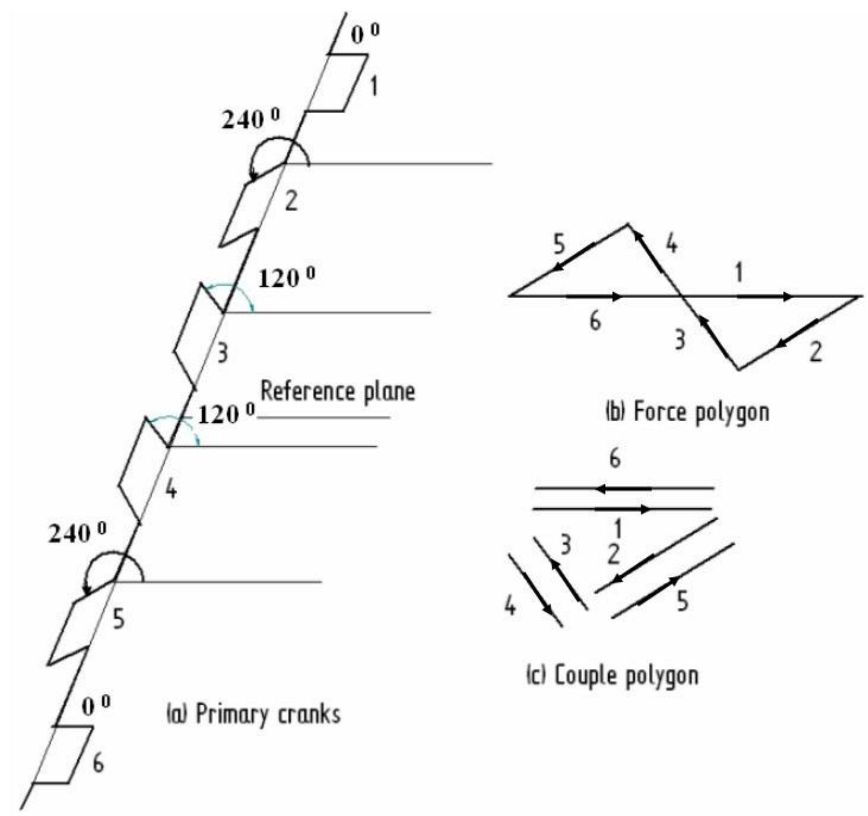
Case 3:

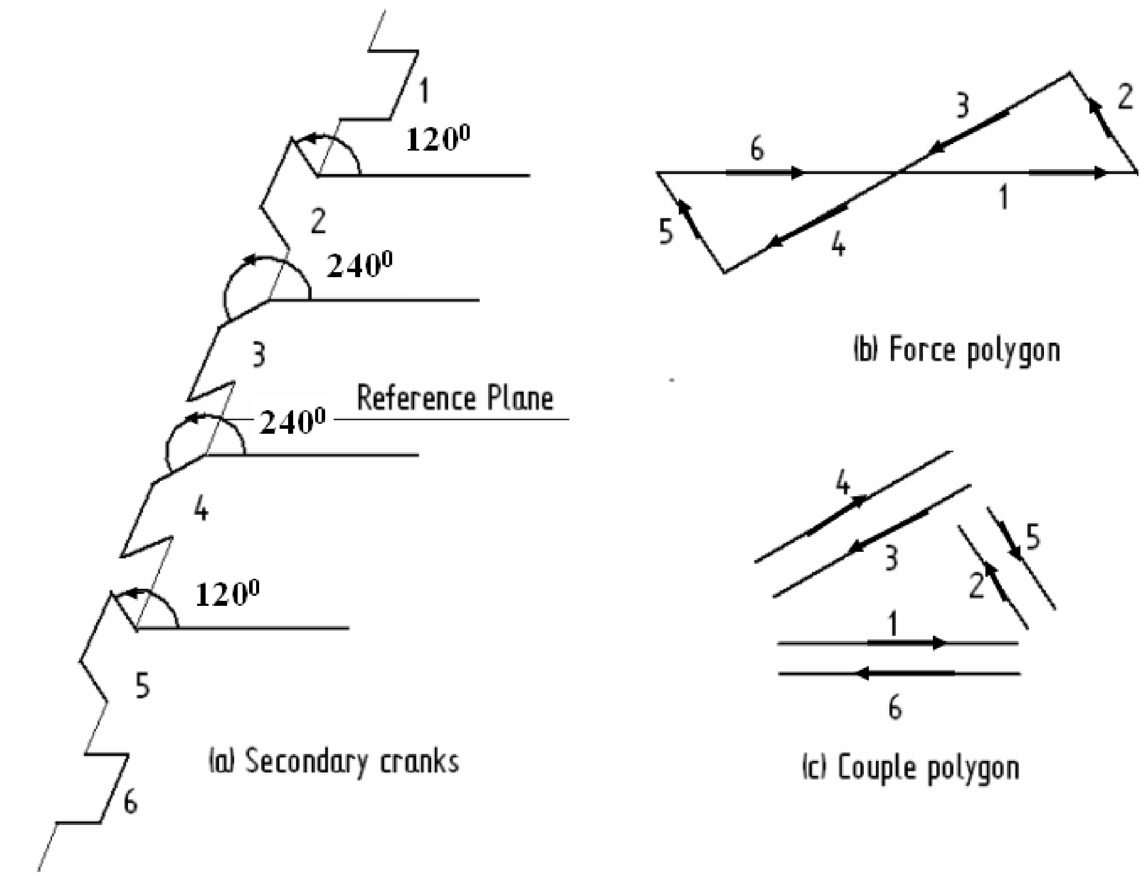
SIX – CYLINDER, FOUR –STROKE ENGINE

Crank positions for different cylinders for the firing order 142635 for clockwise rotation of the crankshaft are, for

First $\theta_1 = 0^0$	Second $\theta_2 = 240^0$	And $m_1 = m_2 = m_3 = m_4 = m_5 = m_6$ $r_1 = r_2 = r_3 = r_4 = r_5 = r_6$
Third $\theta_3 = 120^0$	Fourth $\theta_4 = 120^0$	
Fifth $\theta_5 = 240^0$	Sixth $\theta_6 = 0^0$	

Since all the force and couple polygons close, it is inherently balanced engine for primary and secondary forces and couples.





Problem 1:

Each crank and the connecting rod of a six-cylinder four-stroke in-line engine are 60 mm and 240 mm respectively. The pitch distances between the cylinder centre lines are 80 mm, 80 mm, 100 mm, 80 mm and 80 mm respectively. The reciprocating mass of each cylinder is 1.4 kg. The engine speed is 1000 rpm. Determine the out-of-balance primary and secondary forces and couples on the engine if the firing order be 142635. Take a plane midway between the cylinders 3 and 4 as the reference plane.

Solution:

Given :

$r = 60\,mm$, l = connecting rod length = 240 mm,

m = reciprocating mass of each cylinder = 1.4 kg,

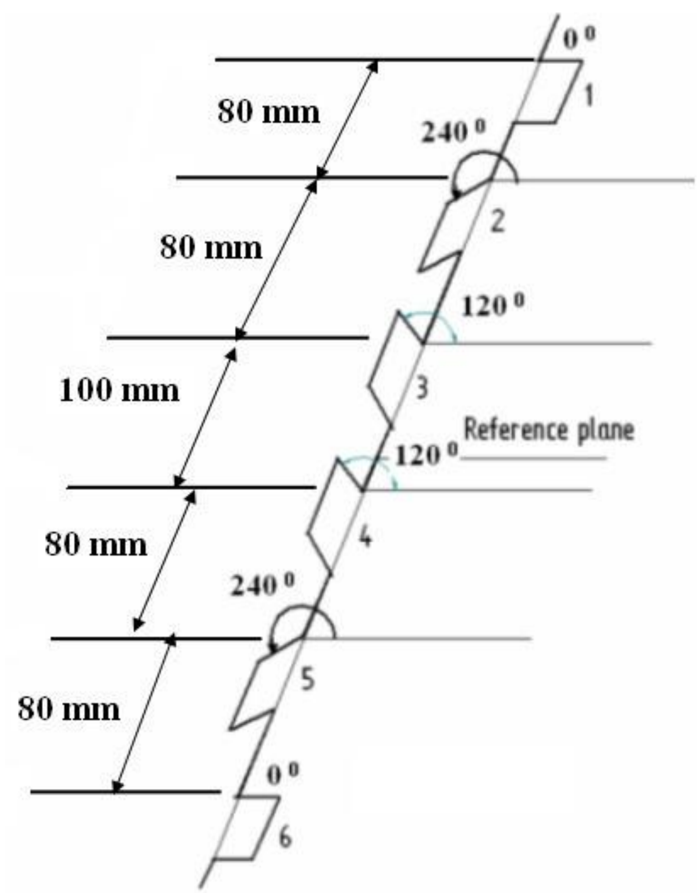
$N = 1000$ rpm

We have, $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 1000}{60} = 104.72$ rad/s

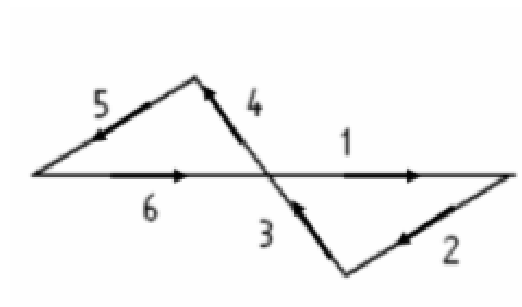
Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane '2' m	Couple/ ω^2 (m r l) kg m^2
1	1.4	0.06	0.084	0.21	0.01764
2	1.4	0.06	0.084	0.13	0.01092
3	1.4	0.06	0.084	0.05	0.0042
4	1.4	0.06	0.084	-0.05	-0.0042
5	1.4	0.06	0.084	-0.13	-0.01092
6	1.4	0.06	0.084	-0.21	-0.01764

Graphical Method:

Step 1:
Draw the primary force and primary couple polygons taking some convenient scales.
Note: For drawing these polygons take primary cranks position as the reference

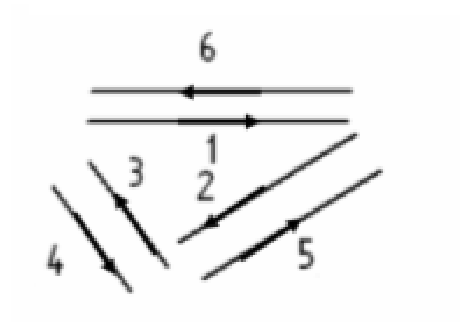


(a) Primary cranks



(b) Force polygon

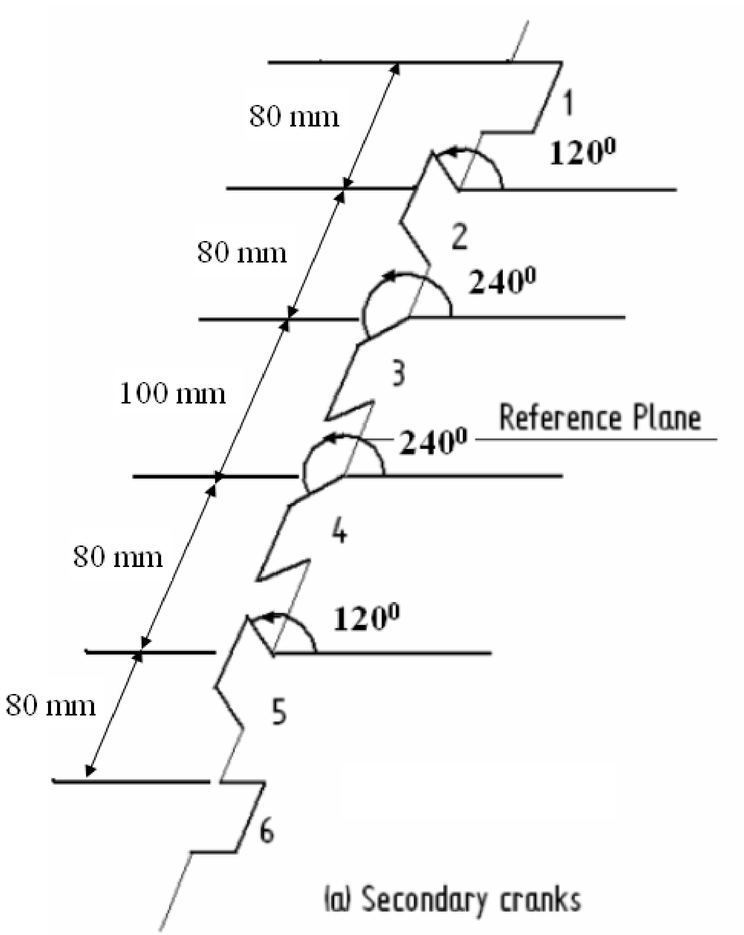
NO UNBALANCED PRIMARY FORCE



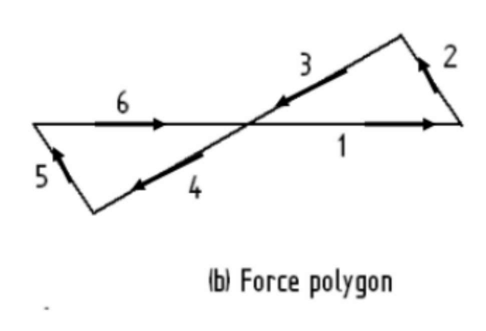
(c) Couple polygon

NO UNBALANCED PRIMARY COUPLE

Step 2:
Draw the secondary force and secondary couple polygons taking some convenient scales.
Note: For drawing these polygons take secondary cranks position as the reference

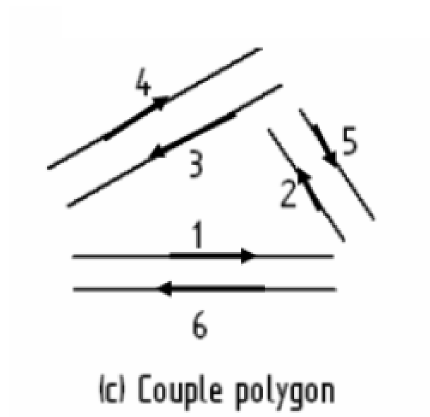


(a) Secondary cranks



(b) Force polygon

NO UNBALANCED SECONDARY FORCE



(c) Couple polygon

NO UNBALANCED SECONDARY COUPLE

Problem 2:

The firing order of a six –cylinder vertical four-stroke in-line engine is 142635. The piston stroke is 80 mm and length of each connecting rod is 180 mm. The pitch distances between the cylinder centre lines are 80 mm, 80 mm, 120 mm, 80 mm and 80 mm respectively. The reciprocating mass per cylinder is 1.2 kg and the engine speed is 2400 rpm. Determine the out-of-balance primary and secondary forces and couples on the engine taking a plane midway between the cylinders 3 and 4 as the reference plane.

Solution:

Given :

$r = \frac{L}{2} = \frac{80}{2} = 40$ mm , l = connecting rod length = 180 mm,

m = reciprocating mass of each cylinder = 1.2 kg,

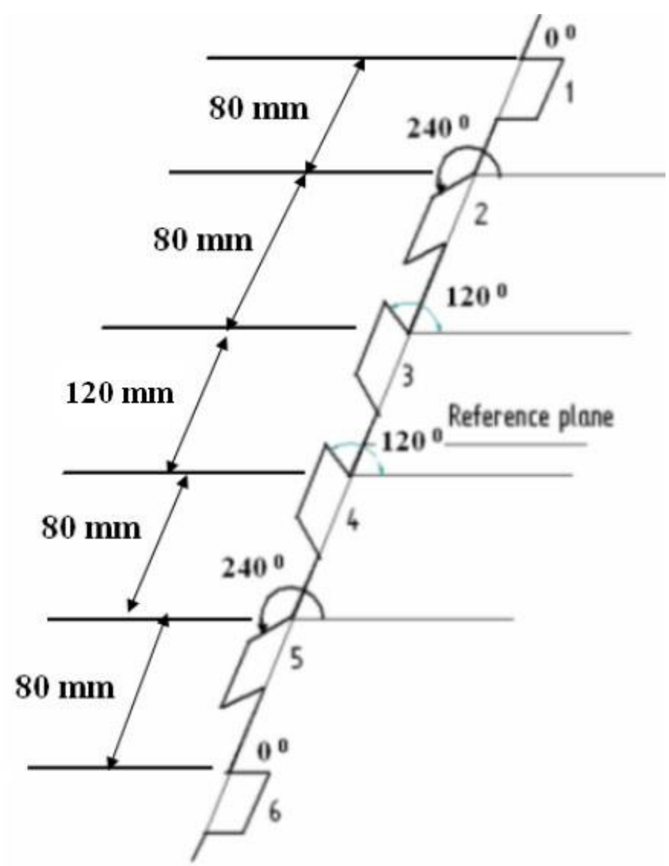
N = 2400 rpm

We have, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 251.33$ rad/s

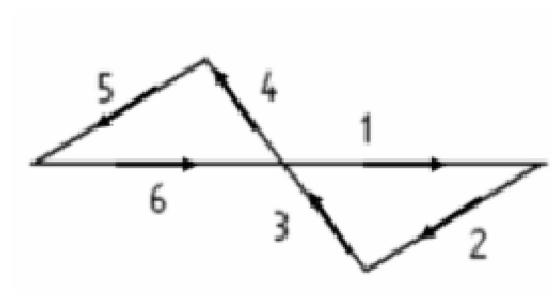
Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane ‘2’ m	Couple/ ω^2 (m r l) kg m^2
1	1.2	0.04	0.048	0.22	0.01056
2	1.2	0.04	0.048	0.14	0.00672
3	1.2	0.04	0.048	0.06	0.00288
4	1.2	0.04	0.048	-0.06	-0.00288
5	1.2	0.04	0.048	-0.14	-0.00672
6	1.2	0.04	0.048	-0.22	-0.01056

Graphical Method:

Step 1:
Draw the primary force and primary couple polygons taking some convenient scales.
Note: For drawing these polygons take primary cranks position as the reference

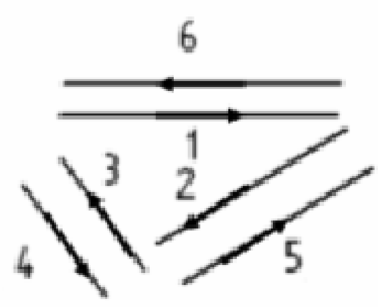


(a) Primary cranks



(b) Force polygon

NO UNBALANCED PRIMARY FORCE



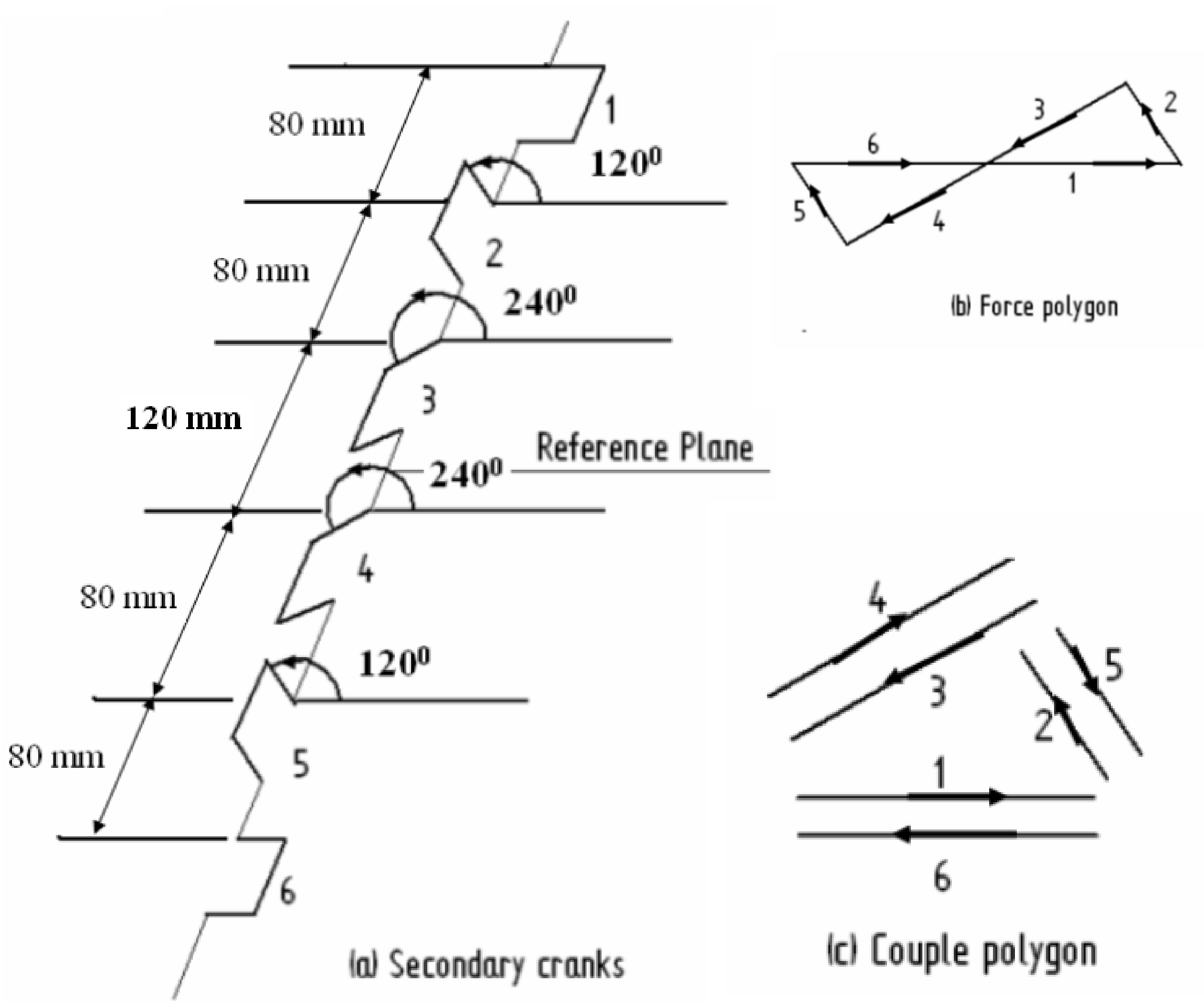
(c) Couple polygon

NO UNBALANCED PRIMARY COUPLE

Step 2:

Draw the secondary force and secondary couple polygons taking some convenient scales.

Note: For drawing these polygons take secondary cranks position as the reference



Problem 3:

The stroke of each piston of a six-cylinder two-stroke inline engine is 320 mm and the connecting rod is 800 mm long. The cylinder centre lines are spread at 500 mm. The cranks are at 60^0 apart and the firing order is 145236. The reciprocating mass per cylinder is 100 kg and the rotating parts are 50 kg per crank. Determine the out of balance forces and couples about the mid plane if the engine rotates at 200 rpm.

Primary cranks position

	Relative positions of Cranks in degrees					
Firing order	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
142635	**0**	**240**	**120**	**120**	**240**	**0**
145236	**0**	**180**	**240**	**60**	**120**	**300**

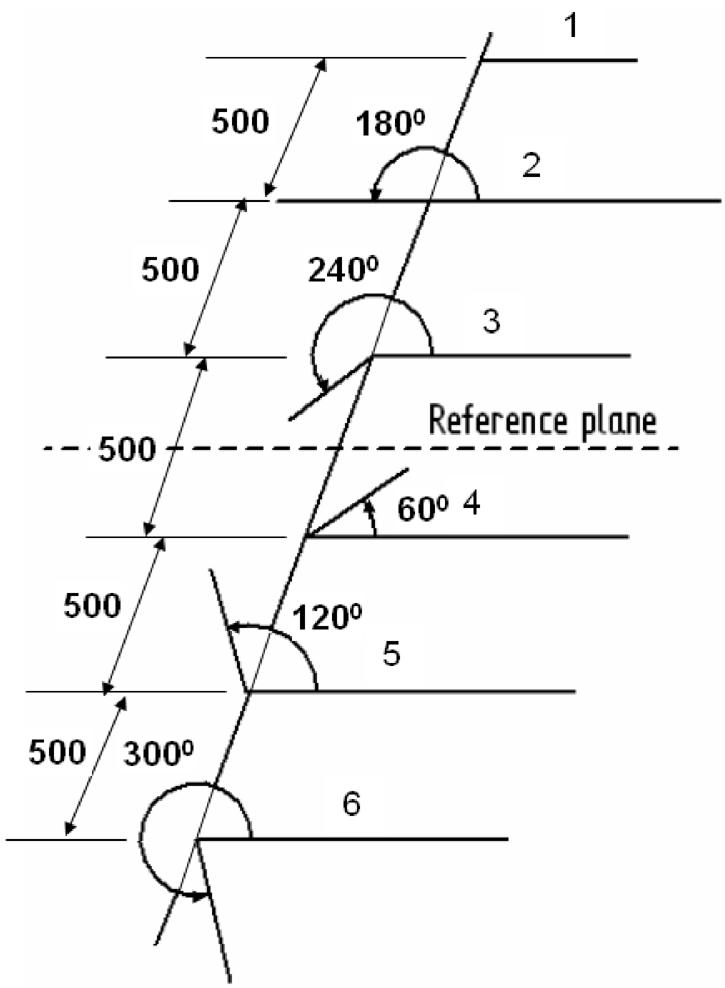
Secondary cranks position

	Relative positions of Cranks in degrees					
Firing order	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
142635	**0**	**120**	**240**	**240**	**120**	**0**
145236	**0**	**0**	**120**	**120**	**240**	**240**

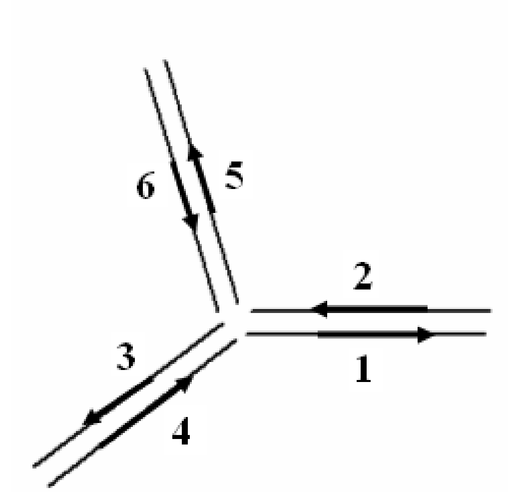
Calculation of primary forces and couples:

Total mass at the crank pin = 100 kg + 50 kg = 150 kg

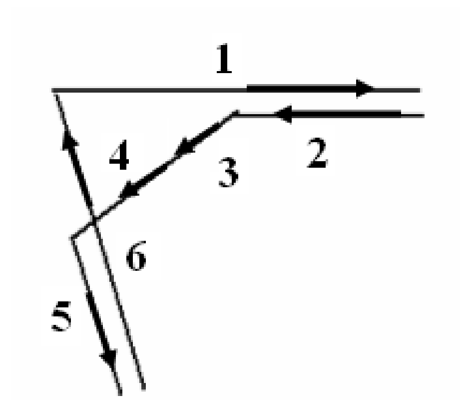
Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane m	Couple/ ω^2 (m r l) kg m^2
1	150	0.16	24	1.25	30
2	150	0.16	24	0.75	18
3	150	0.16	24	0.25	6
4	150	0.16	24	-0.25	-6
5	150	0.16	24	-0.75	-18
6	150	0.16	24	-1.25	-30



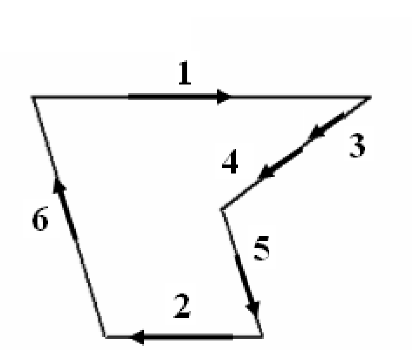
(a) Primary cranks



(b) Force polygon



(c) Couple polygon

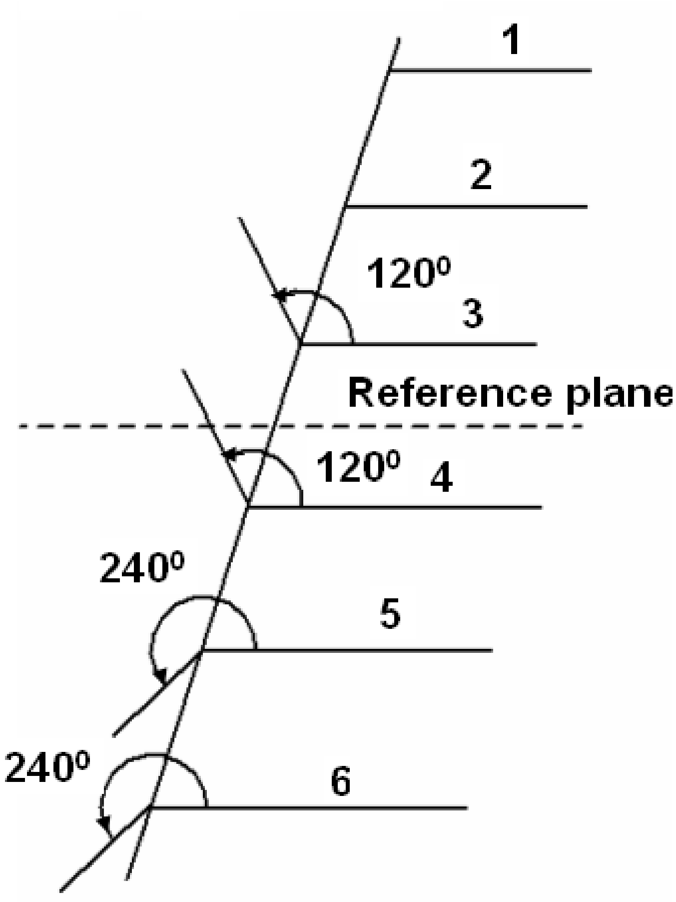


(d) Couple polygon

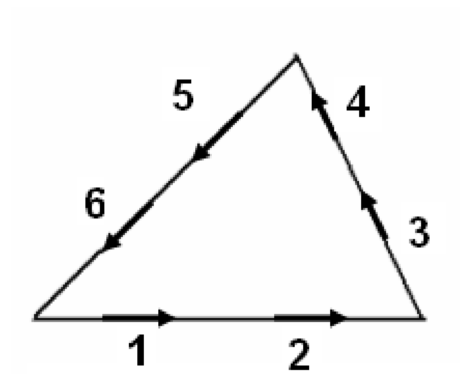
Calculation of secondary forces and couples:

Since rotating mass does not affect the secondary forces as they are only due to second harmonics of the piston acceleration, the total mass at the crank is taken as 100 kg.

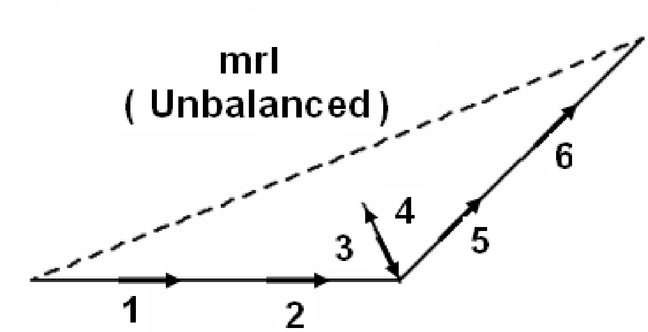
Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane m	Couple/ ω^2 (m r l) kg m^2
1	100	0.16	16	1.25	20
2	100	0.16	16	0.75	12
3	100	0.16	16	0.25	4
4	100	0.16	16	-0.25	-4
5	100	0.16	16	-0.75	-12
6	100	0.16	16	-1.25	-20



(e) Secondary cranks



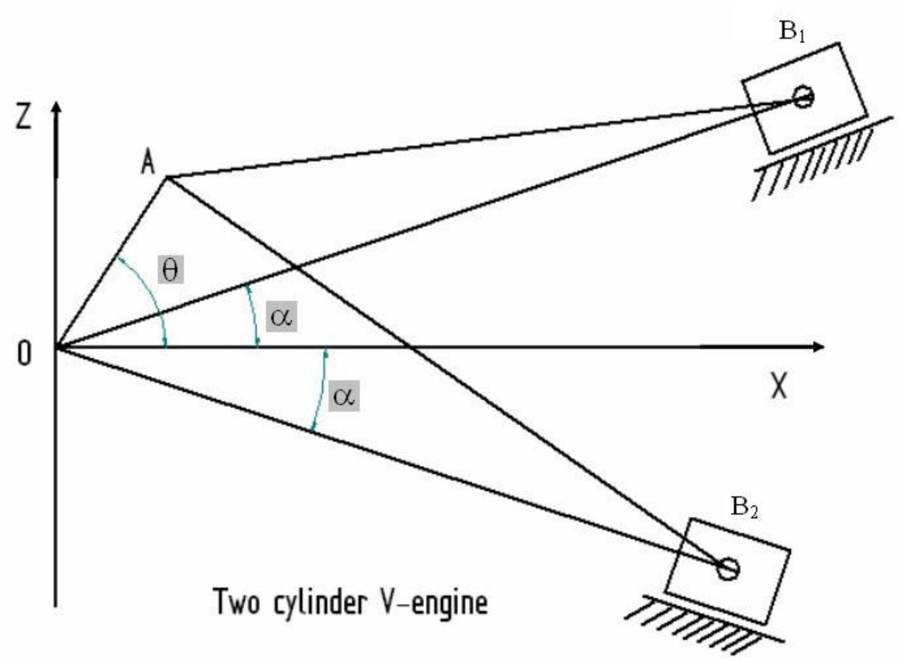
(f) Force polygon



(g) Couple polygon

BALANCING OF V – ENGINE

Two Cylinder V-engine:



A common crank OA is operated by two connecting rods. The centre lines of the two – cylinders are inclined at an angle α to the X-axis.
Let θ be the angle moved by the crank from the X-axis.

Determination of Primary force:

Primary force of 1 along line of stroke OB_1 = $\mathbf{mr}\omega^2 \mathbf{cos(}\theta-\alpha\mathbf{)}------(1)$

Primary force of 1 along X - axis = $\mathbf{mr}\omega^2 \mathbf{cos(}\theta-\alpha\mathbf{)}\ \mathbf{cos}\alpha ---(2)$

Primary force of 2 along line of stroke OB_2 = $\mathbf{mr}\omega^2 \mathbf{cos(}\theta+\alpha\mathbf{)}-----(3)$

Primary force of 2 along X-axis = $\mathbf{mr}\omega^2 \mathbf{cos(}\theta+\alpha\mathbf{)cos}\alpha ---(4)$

Total primary force along X - axis

$$= mr\omega^2 \cos\alpha\,[\cos(\theta-\alpha)+\cos(\theta+\alpha)]$$
$$= mr\omega^2 \cos\alpha\,[\cos\theta\cos\alpha+\sin\theta\sin\alpha+\cos\theta\cos\alpha-\sin\theta\sin\alpha]$$
$$= mr\omega^2 \cos\alpha \times 2\cos\theta\cos\alpha$$
$$= 2\,mr\omega^2\cos^2\alpha\,\cos\theta ------------(5)$$

Similarly,

Total primary force along Z - axis

$$= mr\omega^2 [\cos(\theta - \alpha)\sin\alpha - \cos(\theta + \alpha)\sin\alpha]$$
$$= mr\omega^2 \sin\alpha [(\cos\theta\cos\alpha + \sin\theta\sin\alpha) - (\cos\theta\cos\alpha - \sin\theta\sin\alpha]$$
$$= mr\omega^2 \sin\alpha \times 2\sin\theta\sin\alpha$$
$$= 2\, mr\omega^2 \sin^2\alpha \, \sin\theta ------------(6)$$

Resultant Primary force

$$= \sqrt{(2\, mr\omega^2 \cos^2\alpha \, \cos\theta)^2 + (2\, mr\omega^2 \sin^2\alpha \, \sin\theta)^2}$$
$$= 2\, mr\omega^2 \sqrt{(\cos^2\alpha \, \cos\theta)^2 + (\sin^2\alpha \, \sin\theta)^2} -----(7)$$

and this resultant primary force will be at angle β with the X – axis, given by,

$$\tan\beta = \frac{\sin^2\alpha \, \sin\theta}{\cos^2\alpha \, \cos\theta} ------(8)$$

If $2\alpha = 90^0$, the resultant force will be equal to

$$2\, mr\omega^2 \sqrt{(\cos^2 45^0 \, \cos\theta)^2 + (\sin^2 45^0 \, \sin\theta)^2}$$
$$= mr\omega^2 -----(9)$$

and $$\tan\beta = \frac{\sin^2 45^0 \, \sin\theta}{\cos^2 45^0 \, \cos\theta} = \tan\theta ------(10)$$

i.e., $\beta = \theta$ or it acts along the crank and therefore, can be completely balanced by a mass at a suitable radius diametrically opposite to the crank, such that,

$$m_r r_r = mr -----(11)$$

For a given value of α, the resultant force is maximum (Primary force), when

$$(\cos^2\alpha \, \cos\theta)^2 + (\sin^2\alpha \, \sin\theta)^2 \quad \text{is maximum}$$
$$\text{or}$$
$$(\cos^4\alpha \, \cos^2\theta + \sin^4\alpha \, \sin^2\theta) \quad \text{is maximum}$$

Or

$$\frac{d}{d\theta}\left(\cos^4\alpha\ \cos^2\theta + \sin^4\alpha\ \sin^2\theta\right) = 0$$

i.e., $-\cos^4\alpha \times 2\cos\theta\ \sin\theta + \sin^4\alpha \times 2\sin\theta\cos\theta = 0$

i.e., $-\cos^4\alpha \times \sin 2\theta + \sin^4\alpha \times \sin 2\theta = 0$

i.e., $\sin 2\theta\left[\sin^4\alpha - \cos^4\alpha\right] = 0$

As α is not zero, therefore for a given value of α, the resultant primary force is maximum when $\theta = 0^0$.

Determination of Secondary force:

Secondary force of 1 along line of stroke OB_1 is equal to

$$\frac{mr\omega^2}{n}\cos 2(\theta-\alpha) ------(1)$$

Secondary force of 1 along X - axis $= \dfrac{mr\omega^2}{n}\cos 2(\theta-\alpha)\cos\alpha ---(2)$

Secondary force of 2 along line of stroke OB_2 =

$$\frac{mr\omega^2}{n}\cos 2(\theta+\alpha) -----(3)$$

Primary force of 2 along X-axis $= \dfrac{mr\omega^2}{n}\cos 2(\theta+\alpha)\cos\alpha ---(4)$

Therefore,

Total secondary force along X-axis

$$= \frac{mr\omega^2}{n}\cos\alpha\left[\cos 2(\theta-\alpha) + \cos 2(\theta+\alpha)\right]$$

$$= \frac{mr\omega^2}{n}\cos\alpha\left[(\cos 2\theta\cos 2\alpha + \sin 2\theta\sin 2\alpha) + (\cos 2\theta\cos 2\alpha - \sin 2\theta\sin 2\alpha\right]$$

$$= \frac{2mr\omega^2}{n}\cos\alpha\ \cos 2\theta\cos 2\alpha ------------(5)$$

Similarly,

Total secondary force along Z-axis

$$= \frac{2mr\omega^2}{n} \sin\alpha \ \sin 2\theta \sin 2\alpha ----------(6)$$

Resultant Secondary force

$$= \frac{2mr\omega^2}{n} \sqrt{(\cos\alpha \ \cos 2\theta \cos 2\alpha)^2 + (\sin\alpha \ \sin 2\theta \sin 2\alpha)^2} -----(7)$$

And $\tan\beta' = \dfrac{\sin\alpha \ \sin 2\theta \sin 2\alpha}{\cos\alpha \ \cos 2\theta \cos 2\alpha}$ ------(8)

If $2\alpha = 90^0$ or $\alpha = 45^0$,

Secondary force = $\dfrac{2mr\omega^2}{n}\sqrt{\left(\dfrac{\sin 2\theta}{\sqrt{2}}\right)^2} = \sqrt{2}\,\dfrac{mr\omega^2}{n}\sin 2\theta$ ----(9)

And $\tan\beta' = \infty$ and $\beta' = 90^0$ ------(10) i.e., the force acts along Z-axis and is a harmonic force and special methods are needed to balance it.

Problem 1:

The cylinders of a twin V-engine are set at 60^0 angle with both pistons connected to a single crank through their respective connecting rods. Each connecting rod is 600 mm long and the crank radius is 120 mm. The total rotating mass is equivalent to 2 kg at the crank radius and the reciprocating mass is 1.2 kg per piston. A balance mass is also fitted opposite to the crank equivalent to 2.2 kg at a radius of 150 mm. Determine the maximum and minimum values of the primary and secondary forces due to inertia of the reciprocating and the rotating masses if the engine speed is 800 mm.

Solution:

Given:

m = reciprocating mass of each piston = 1.2 kg

M = equivalent rotating mass = 2 kg

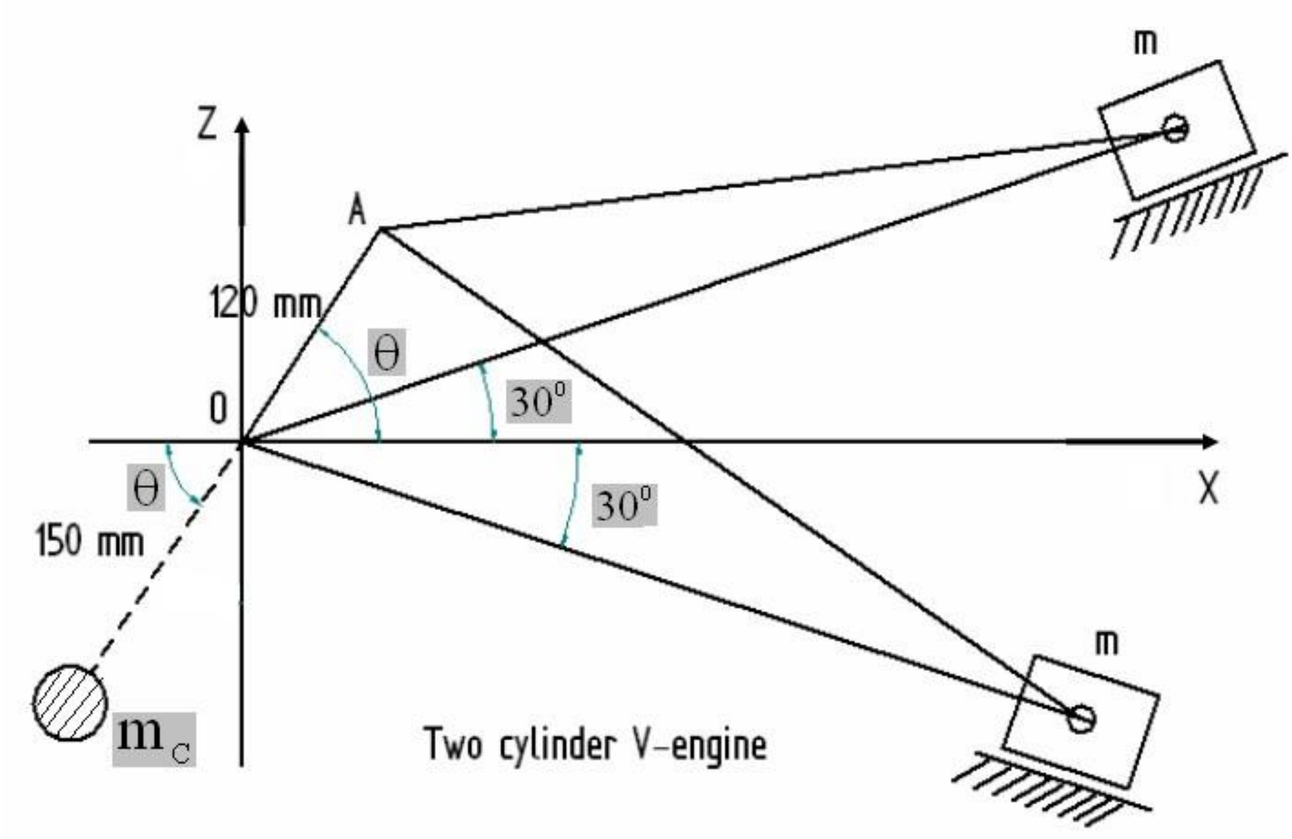
m_c = balancing mass = 2.2 kg, r_c = 150 mm

l = connecting rod length = 600 mm

r = crank radius = 120 mm

N = 800 rpm

We have, $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 800}{60} = 83.78$ rad/s and $n = \dfrac{l}{r} = \dfrac{600}{120} = 5$



Total primary force along X - axis $= 2\, m\, r\, \omega^2 \cos^2 \alpha \cos\theta$ ------------(1)

Centrifugal force due to rotating mass along X – axis

$$= M\, r\, \omega^2 \cos\theta \text{ ------------(2)}$$

Centrifugal force due to balancing mass along X – axis

$$= -m_c\, r_c\, \omega^2 \cos\theta \text{ ------------(3)}$$

Therefore total unbalance force along X –axis = (1) + (2) + (3)

That is

Total Unbalance force along X axis

$$= 2\, m r \omega^2 \cos^2 \alpha \cos\theta + M r \omega^2 \cos\theta - m_c r_c \omega^2 \cos\theta$$

$$= \omega^2 \cos\theta \left[2\, m r \cos^2\alpha + M r - m_c r_c\right]$$

$$= (83.78)^2 \cos\theta \left[2x1.2x0.12x\cos^2 30^0 + 2x0.12 - 2.2x0.15\right]$$

$$= (83.78)^2 \cos\theta \left[0.216 + 0.24 - 0.33\right] = 884.41 \cos\theta \text{ N -------(4)}$$

Total primary force along Z - axis $= 2\, m\, r\, \omega^2 \sin^2 \alpha \sin\theta$ ------------(5)

Centrifugal force due to rotating mass along Z – axis

$$= M r \omega^2 \sin\theta ------------(6)$$

Centrifugal force due to balancing mass along Z – axis

$$= -m_c r_c \omega^2 \sin\theta ------------(7)$$

Therefore total unbalance force along Z –axis = (5) + (6) + (7)

That is

Total Unbalance force along Z - axis

$$= 2\, m r \omega^2 \sin^2\alpha\, \sin\theta + M r \omega^2 \sin\theta - m_c r_c \omega^2 \sin\theta$$

$$= \omega^2 \sin\theta \left[2\, m r \sin^2\alpha + M r - m_c r_c\right]$$

$$= (83.78)^2 \sin\theta \left[2 x 1.2 x 0.12 x \sin^2 30^0 + 2 x 0.12 - 2.2 x 0.15\right]$$

$$= (83.78)^2 \sin\theta \left[0.072 + 0.24 - 0.33\right] = -126.34 \sin\theta\ N -------(8)$$

Resultant Primary force

$$= \sqrt{(884.41 \cos\theta)^2 + (-126.34 \sin\theta)^2}$$

$$= \sqrt{782181.05 \cos^2\theta + 15961.8 \sin^2\theta}$$

$$= \sqrt{766219.25 \cos^2\theta + 15961.8} -----(9)$$

This is maximum, when $\theta = 0^0$ and minimum, when $\theta = 90^0$

Maximum Primary force, i.e., when $\theta = 0^0$

$$= \sqrt{766219.25 + 15961.8} = 884.41\ N -----(10)$$

And Minimum Primary force, i.e., when $\theta = 90^0$

$$= \sqrt{766219.25 \cos^2 90^0 + 15961.8} = 126.34\ N -----(11)$$

Secondary force:

The rotating masses do not affect the secondary forces as they are only due to second harmonics of the piston acceleration.

Resultant Secondary force

$$= \frac{2mr\omega^2}{n}\sqrt{(\cos\alpha \;\; \cos 2\theta \cos 2\alpha)^2 + (\sin\alpha \;\; \sin 2\theta \sin 2\alpha)^2}$$

$$= \frac{2 x 1.2 x 0.12 x (83.78)^2}{5}\sqrt{\begin{array}{l}(\cos 30^2 \;\; \cos 2\theta \cos 60^2)^2 \\ +(\sin 30^0 \;\; \sin 2\theta \sin 60^0)^2\end{array}}$$

$$= 404.3\sqrt{\left[0.1875\,(\cos 2\theta)^2 + 0.1875\,(\sin 2\theta)^2\right]} -----(12)$$

This is maximum, when $\theta = 0^0$ and minimum, when $\theta = 180^0$

Maximum secondary force, i.e., when $\theta = 0^0$

$$= 404.3\sqrt{\left[0.1875\,(\cos 0^0)^2 + 0.1875\,(\sin 0^0)^2\right]} - = 175.07 \text{ N} -----(13)$$

And Minimum secondary force, i.e., when $\theta = 180^0$

$$= 404.3\sqrt{\left[0.1875\,(\cos 180^0)^2 + 0.1875\,(\sin 180^0)^2\right]} = 175.07 \text{ N} -----(14)$$

BALANCING OF W, V-8 AND V-12 – ENGINES

BALANCING OF W ENGINE

In this engine three connecting rods are operated by a common crank.

Total primary force along X - axis

$$= mr\omega^2 \cos\theta\left(2\cos^2\alpha + 1\right) ------------(1)$$

Total primary force along Z-axis will be same as in the V – twin engine, (since the primary force of 3 along Z – axis is zero)

$$= 2\, mr\omega^2 \sin^2\alpha \;\; \sin\theta ------------(2)$$

Resultant Primary force

$$= mr\omega^2\sqrt{\left[\cos\theta\left(2\cos^2\alpha + 1\right)^2 + \left(2\sin^2\alpha\sin\theta\right)^2\right]} -----(3)$$

and this resultant primary force will be at angle β with the X – axis, given by,

$$\tan\beta = \frac{2\sin^2\alpha\ \sin\theta}{\cos\theta\left(2\cos^2\alpha+1\right)} ------(4)$$

If $\alpha = 60^0$,

Resultant Primary force

$$= \frac{3}{2}mr\omega^2 -----(5)$$

and

$$\tan\beta = \ = \tan\theta ------(6)$$

i.e., $\beta = \theta$ or it acts along the crank and therefore, can be completely balanced by a mass at a suitable radius diametrically opposite to the crank, such that, $m_r r_r = mr$ -----(7)

Total secondary force along X-axis

$$= \cos 2\theta\left[\frac{2mr\omega^2}{n}\cos\alpha\cos 2\alpha+1\right] ------------(8)$$

Total secondary force along Z –direction will be same as in the V-twin engine.

Resultant secondary force

$$= \frac{mr\omega^2}{n}\sqrt{\left[\cos 2\theta\left(2\cos\alpha\cos 2\alpha+1\right)^2+\left(2\sin\alpha\sin 2\alpha\sin 2\theta\right)^2\right]} -----(9)$$

$$\tan\beta' = \frac{2\sin\alpha\ \sin 2\theta\sin 2\alpha}{\cos 2\theta\left(2\cos\alpha\cos 2\alpha+1\right)} ------(10)$$

If $\alpha = 60^0$,

Secondary force along X-axis

$$= \frac{mr\omega^2}{2n}\cos 2\theta ------------(11)$$

Secondary force along Z-axis

$$= \frac{3mr\omega^2}{2n}\sin 2\theta ------------(12)$$

It is not possible to balance these forces simultaneously

V-8 ENGINE

It consists of two banks of four cylinders each. The two banks are inclined to each other in the shape of V. The analysis will depend on the arrangement of cylinders in each bank.

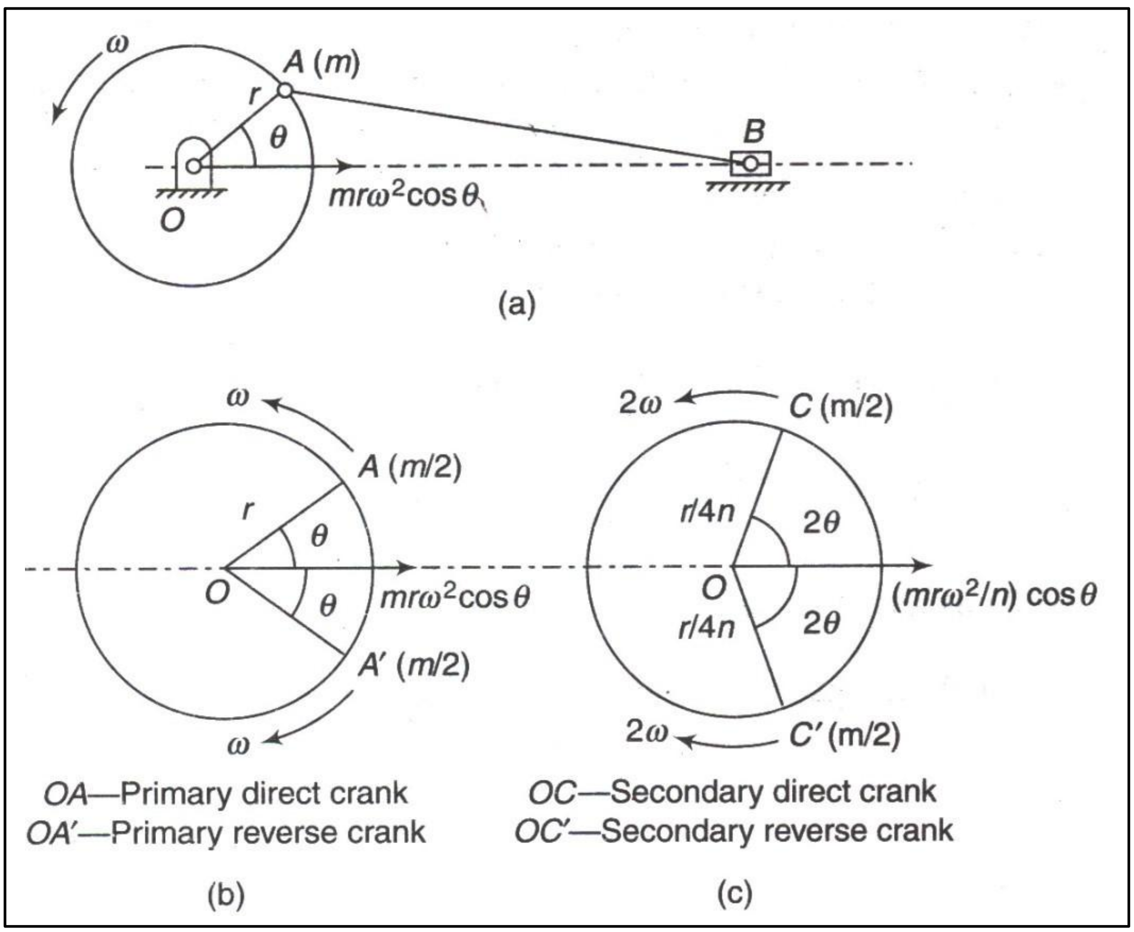
V-12 ENGINE

It consists of two banks of six cylinders each. The two banks are inclined to each other in the shape of V. The analysis will depend on the arrangement of cylinders in each bank.

If the cranks of the six cylinders on one bank are arranged like the completely balanced six cylinder, four stroke engine then, there is no unbalanced force or couple and thus the engine is completely balanced.

BALANCING OF RADIAL ENGINES:

It is a multicylinder engine in which all the connecting rods are connected to a common crank.



Direct and reverse crank method of analysis:

In this all the forces exists in the same plane and hence no couple exist.

In a reciprocating engine the primary force is given by, $\mathbf{m}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\,\theta$ which acts along the line of stroke.

In direct and reverse crank method of analysis, a force identical to this force is generated by two masses as follows.

1.A mass m/2, placed at the crank pin A and rotating at an angular velocity ω in the counter clockwise direction.
2.A mass m/2, placed at the crank pin of an imaginary crank OA' at the same angular position as the real crank but in the opposite direction of the line of stroke. It is assumed to rotate at an angular velocity ω in the clockwise direction (opposite).
3. While rotating, the two masses coincide only on the cylinder centre line.

The components of the centrifugal forces due to rotating masses along the line of stroke are,

$$\textbf{Due to mass at } \mathbf{A} = \frac{\mathbf{m}}{2}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\,\theta$$

$$\textbf{Due to mass at } \mathbf{A}' = \frac{\mathbf{m}}{2}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\,\theta$$

Thus, total force along the line of stroke = $\mathbf{m}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\,\theta$ which is equal to the primary force.
At any instant, the components of the centrifugal forces of these masses normal to the line of stroke will be equal and opposite.
The crank rotating in the direction of engine rotation is known as the **direct crank** and the imaginary crank rotating in the opposite direction is known as the **reverse crank.**

Now,

Secondary accelerating force is

$$\mathbf{m}\,\mathbf{r}\,\omega^2\,\frac{\mathbf{cos}\,2\theta}{\mathbf{n}} = \mathbf{m}\,\mathbf{r}\,(2\omega)^2\,\frac{\mathbf{cos}\,2\theta}{4\mathbf{n}}$$

$$= \mathbf{m}\,\frac{\mathbf{r}}{4\mathbf{n}}(2\omega)^2\,\mathbf{cos}\,2\theta$$

This force can also be generated by two masses in a similar way as follows.

1. A mass m/2, placed at the end of direct secondary crank of length $\frac{\mathbf{r}}{4\mathbf{n}}$ at an angle 2θ and rotating at an angular velocity 2ω in the counter clockwise direction.

2. A mass m/2, placed at the end of reverse secondary crank of length $\frac{\mathbf{r}}{4\mathbf{n}}$ at an angle -2θ and rotating at an angular velocity 2ω in the clockwise direction.

The components of the centrifugal forces due to rotating masses along the line of stroke are,

$$\text{Due to mass at } C = \frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{2n}\cos 2\theta$$

$$\text{Due to mass at } C' = \frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{2n}\cos 2\theta$$

Thus, total force along the line of stroke =

$2\mathbf{x}\frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{n}\cos 2\theta$ which is equal to the secondary force.

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